Subatomic Physics
Solutions 2018
Open Questions

April 10, 2018

1 Part 2 - Open questions

1.1 Rutherford Scattering

Solutions

a) See Figure 1.

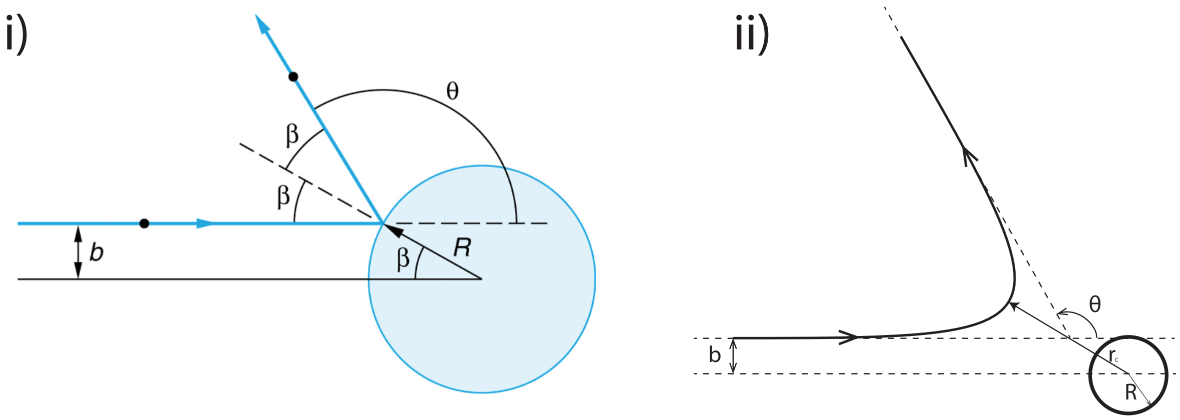


Figure 1: i) Diagram should show scattering at approximately correct angle and with angle of incidence approximately equal to angle of reflection. b and θ must be indicated and defined as scattering parameter and scattering angle respectively. β and R are not required. ii) Path should have smooth curve resembling hyperbola at approximately correct angle. b and θ must be marked. r_C and R are not required.

b) Equation **??** will hold as long as $r_0 > R$. This corresponds to having a low enough incident energy, low enough R and high enough charges on the two particles. The conditions at departure form Rutherford law allow a measurement of the nuclear radius.

c) i) First will need expression for b for the hard sphere. From Exercise Sheet 1 solutions:

$$b = R\sin\beta = R\sin\left(\frac{180^\circ - \theta}{2}\right)$$
$$= R\cos\frac{\theta}{2}.$$

To calculate b, the radius is required. This may be calculated using $R = r_0 A^{1/3} = 1.25\,\text{fm} \times 208^{1/3} = 7.41\,\text{fm}$.

This yields $b_{20^\circ} = 7.30\,\text{fm}$. The relevant cross section is the circle with radius b:

$$\sigma = \pi b^2 = 167\,(\text{fm})^2$$

ii) For the charged sphere, the expression to calculate b is given:

$$b_{20^\circ} = \frac{1}{4\pi\epsilon_0}\frac{ZZ'e^2}{2E}\cot\frac{\theta}{2} = 1.44\text{eVnm} \times \frac{1\times 82}{2\times 10\times 10^6}\cot\frac{10^\circ}{2} = 3.35\times 10^{-5}\,\text{nm} = 33.4\,\text{fm}$$

$$\sigma = \pi b^2 = 3522\,(\text{fm})^2$$

d) Realise that the particles scattered into $\mathrm{d}\theta \to \theta + \mathrm{d}\theta$ arrive inside the transverse ring bounded by $b \to b - \mathrm{d}b$ with area $2\pi b\mathrm{d}b$. This is the same area as is described by Equation **??**. Therefore:

$$\Delta\sigma(\theta,\phi) = -\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}(\theta,\phi)\sin\theta\mathrm{d}\theta\mathrm{d}\phi = 2\pi b\mathrm{d}b.$$

The scattering is symmetric about ϕ so this dependence may already be integrated, cancelling the 2π factor. Hence:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = -\frac{b}{\sin\theta}\frac{\mathrm{d}b}{\mathrm{d}\theta}$$

This is the required expression which would allow calculation of the partial cross section given $b(\theta)$.

1.2 Direct detection of dark matter

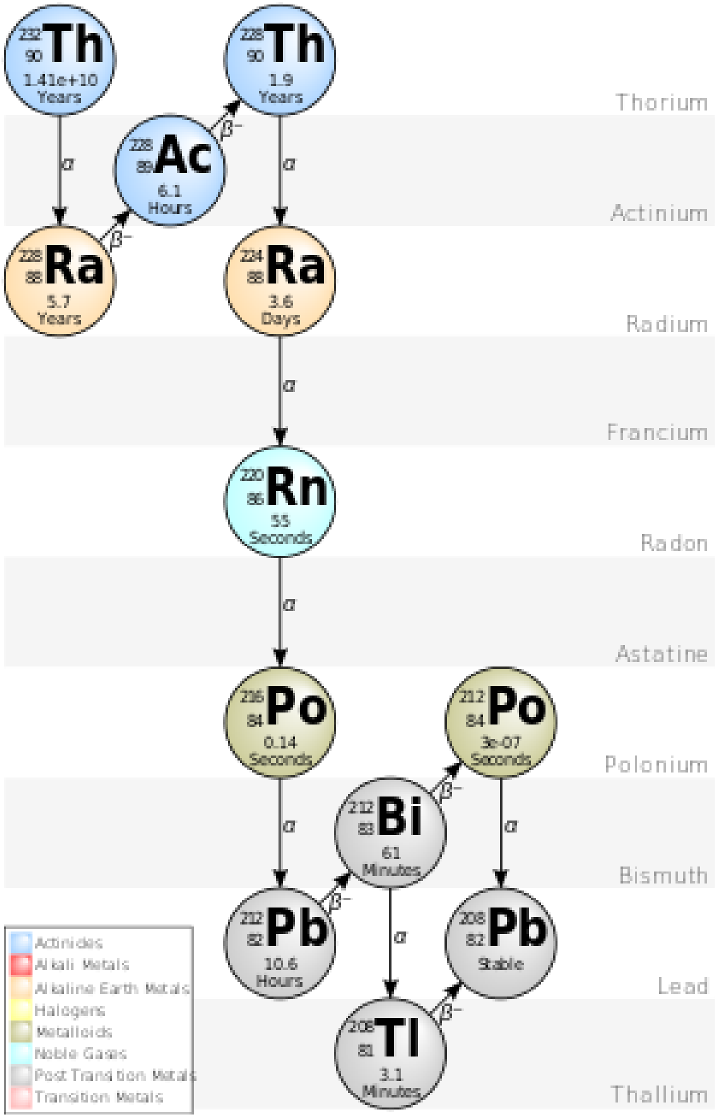


Figure 2: Decay chain of ^{232}Th.

Solutions

a) A WIMP is approaching the nucleus (at rest) with a non-relativistic speed v. In the center-of-momentum frame, both the WIMP and the nucleus have momentum $p\prime$ (with respectively speeds $v\prime$ and $V\prime$). So we have the relations:

$$v = v\prime + V\prime = (\frac{1}{m_\chi} + \frac{1}{M_N})p = \frac{m_\chi + M_N}{m_\chi M_N} \tag{1}$$

$$p = \frac{m_\chi M_N}{m_\chi + M_N} v = \mu v \tag{2}$$

The WIMP and nucleus scattering with an angle θ, see Figure 3. This means:

$$\vec{p} = (\mu v, 0, 0) \tag{3}$$
$$\vec{p\prime} = (\mu v \cos\theta, \mu v \sin\theta) \tag{4}$$
$$\vec{q} = \vec{p\prime} - \vec{p} = (\mu v(\cos\theta - 1), \mu v \sin\theta, 0) \tag{5}$$

The recoil energy is given by:

$$E_R = \frac{1}{2} M_N V_R^2 = \frac{|\vec{p_R}|^2}{2M_N} = \frac{|\vec{q}|^2}{2M_N} \tag{6}$$

and

$$|\vec{q}|^2 = \mu^2 v^2 \left((\cos\theta - 1)^2 + \sin\theta^2\right) = 2\mu^2 v^2 (1 - \cos\theta) \tag{7}$$

So

$$E_R = \frac{\mu^2 v^2}{M_N}(1 - \cos\theta) \tag{8}$$

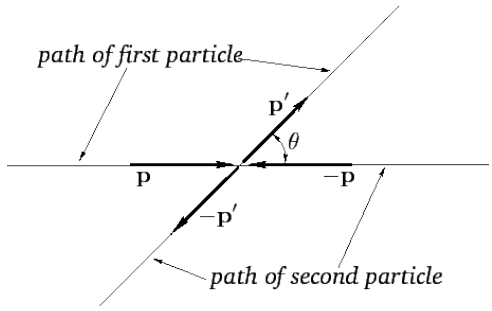


Figure 3: Scattering of WIMP on nucleus in center-of-momentum frame

b) Plugging in the given values for m_χ, v and using $M_N \approx 131u \approx 131$ GeV$/c^2$, we find:

$$E_R = \frac{m_\chi^2 M_N v^2}{(m_\chi + M_N)^2}(1 - \cos\theta) \approx 26\,\text{keV}. \tag{9}$$

So a typical energy scale of several keV.

c) Following the definition of the cross section (*"their mutual cross section is the area transverse to their relative motion within which they must meet in order to scatter from each other"*), the number of expected events (per unit time) is just given by: $N_{\text{events}} = N_{\text{targets}} \Phi \sigma_N$.

d) The relevant decays are $^{232}\text{Th} \to {}^{228}\text{Ra} + \alpha$ and $^{228}\text{Ra} \to {}^{228}\text{Ac} + \beta^-$. The rate of change of ^{228}Ra is therefore given by:

$$\frac{\mathrm{dN}_{^{228}\mathrm{Ra}}(\mathrm{t})}{\mathrm{dt}} = -\lambda_{^{228}\mathrm{Ra}}\ N_{^{228}\mathrm{Ra}}(t) + \lambda_{^{232}\mathrm{Th}}\ N_{^{232}\mathrm{Th}}(t) \tag{10}$$

where we used for $N_{^{232}\mathrm{Th}}(t)$ the solution for exponential decay.

e) Using the solution for exponential decay

$$\frac{\mathrm{dN}_{^{232}\mathrm{Th}}(\mathrm{t})}{\mathrm{dt}} = -\lambda_{^{232}\mathrm{Th}}\ N_{^{232}\mathrm{Th}}(t)$$
$$N_{^{232}\mathrm{Th}}(t) = N_{^{232}\mathrm{Th}}(0)\ e^{-\lambda_{^{232}\mathrm{Th}} t} \tag{11}$$

We can rewrite equation 10.

$$\begin{aligned}\frac{\mathrm{dN}_{^{228}\mathrm{Ra}}(\mathrm{t})}{\mathrm{dt}} &= -\lambda_{^{228}\mathrm{Ra}}\ N_{^{228}\mathrm{Ra}}(t) + \lambda_{^{232}\mathrm{Th}}\ N_{^{232}\mathrm{Th}}(t) \\ &= -\lambda_{^{228}\mathrm{Ra}}\ N_{^{228}\mathrm{Ra}}(t) + \lambda_{^{232}\mathrm{Th}}\ N_{^{232}\mathrm{Th}}(0)\ e^{-\lambda_{^{232}\mathrm{Th}} t}\end{aligned} \tag{12}$$

Shuffling and multiply with $e^{+\lambda_{^{228}\mathrm{Ra}} t}$, we find:

$$e^{\lambda_{^{228}\mathrm{Ra}} t}\ \frac{\mathrm{dN}_{^{228}\mathrm{Ra}}(\mathrm{t})}{\mathrm{dt}} + e^{\lambda_{^{228}\mathrm{Ra}} t}\ \lambda_{^{228}\mathrm{Ra}}\ N_{^{228}\mathrm{Ra}}(t) = e^{\lambda_{^{228}\mathrm{Ra}} t}\ \lambda_{^{232}\mathrm{Th}}\ N_{^{232}\mathrm{Th}}(0)\ e^{-\lambda_{^{232}\mathrm{Th}} t} \tag{13}$$

which is the same as

$$\frac{\mathrm{d}}{\mathrm{dt}}\left[e^{\lambda_{^{228}\mathrm{Ra}} t}\ N_{^{228}\mathrm{Ra}}(t)\right] = \lambda_{^{232}\mathrm{Th}}\ N_{^{232}\mathrm{Th}}(0)\ e^{(\lambda_{^{228}\mathrm{Ra}} - \lambda_{^{232}\mathrm{Th}})t} \tag{14}$$

Integrate over time, $\int_0^t$:

$$e^{\lambda_{^{228}\mathrm{Ra}} t}\ N_{^{228}\mathrm{Ra}}(t) - N_{^{228}\mathrm{Ra}}(0) = \frac{\lambda_{^{232}\mathrm{Th}}\ N_{^{232}\mathrm{Th}}(0)}{\lambda_{^{228}\mathrm{Ra}} - \lambda_{^{232}\mathrm{Th}}}(e^{(\lambda_{^{228}\mathrm{Ra}} - \lambda_{^{232}\mathrm{Th}})t} - 1) \tag{15}$$

Assuming there was no 228-Ra from the start, and the definition of activity $A(t) = \lambda N(t)$, we find equation **??**.

$$A_{^{228}\mathrm{Ra}}(t) = A_{^{232}\mathrm{Th}}(0)\frac{\lambda_{^{228}\mathrm{Ra}}}{\lambda_{^{228}\mathrm{Ra}} - \lambda_{^{232}\mathrm{Th}}}(e^{-\lambda_{^{232}\mathrm{Th}} t} - e^{-\lambda_{^{228}\mathrm{Ra}} t}) \tag{16}$$

f) If we look at the lifetimes in Figure 2, we see that for every reasonable timescale for an experiment, $\lambda_{^{232}\mathrm{Th}} = 1/t_{1/2,^{232}\mathrm{Th}} = 1/\mathrm{large} \approx 0$. So equation **??** simplifies to

$$A_{^{228}\mathrm{Ra}}(t) \approx A_{^{232}\mathrm{Th}}(0)\ (1 - e^{-\lambda_{^{228}\mathrm{Ra}} t}) \tag{17}$$

g) See figure 4.

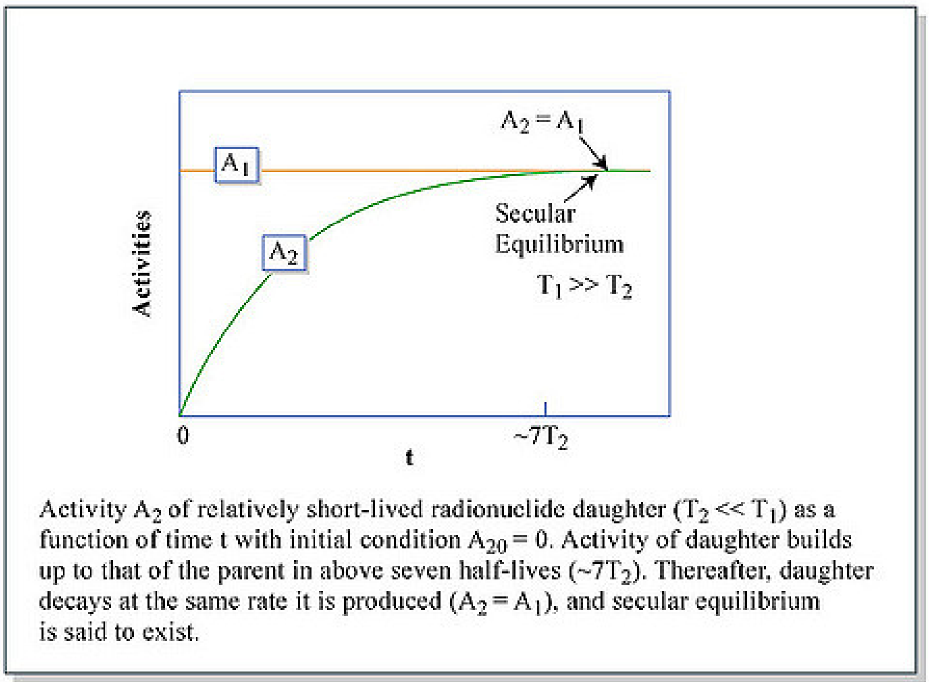


Figure 4: Solution to d).

h) After long enough time (but still $t \ll t_{1/2,Th}$), all isotopes in the decay chain will be in secular equilibrium, so all rates of change are zero: $\frac{\mathrm{dN_d(t)}}{\mathrm{dt}} = 0$. Therefore, $\lambda_d\ N_d(t) = \lambda_p\ N_p(t)$ (where d = Daughter, and p = Parent). Therefore, if we measure the amount of one isotope at time t, we know all amounts. Example:

$$
\begin{aligned}
{}^{216}\mathrm{Po}: \qquad \mathrm{N}_{^{216}\mathrm{Po}} &= \frac{\lambda_{^{220}\mathrm{Rn}}}{\lambda_{^{216}\mathrm{Po}}}\ N_{^{220}\mathrm{Rn}} \\
&= \frac{1/55\ \mathrm{s}^{-1}}{1/0.14\ \mathrm{s}^{-1}}\ N_{^{220}\mathrm{Rn}} \qquad (18)
\end{aligned}
$$

i) Neutrinos. They can undergo a weak interaction with one of the atomic electrons, causing electronic recoil. Just as for the other electronic recoils these may leak into the nuclear recoil signal region, if there are enough of them. A second way in which neutrinos **may** leave a signal in your DM detector is by coherent scattering to the atomic nucleus (not yet observed). For all the other Standard Model particles you can shield your detector.

1.3 Conservation laws and Feynman diagrams

Solutions

a) $\mu^- \to \overline{\nu_\mu} + e^- + \overline{\nu_e}$

- **Not allowed**, lepton number is not conserved

b) $\pi^0 + p \to n + p$

- **Not allowed**, baryon number is not conserved

c) $e^- + p \to n + \nu_e$

- **Allowed**, electron capture using weak interaction

d) $D^+ \to k^- + \pi^+ + \pi^+$

- **Allowed**, weak decay (charm to strange) and quark pair creation using a gluon

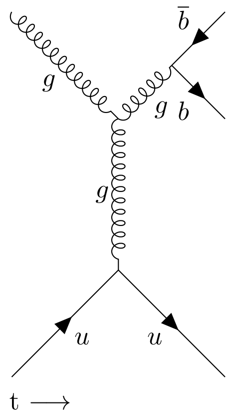
e) $p \to p + k^- + k^+$

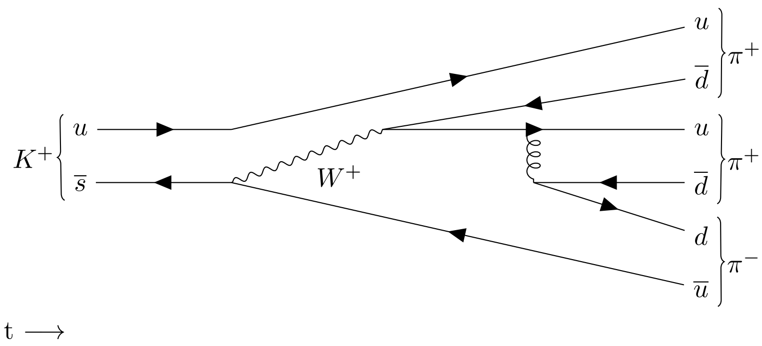
- **Not allowed**, energy is not conserved

f) $J/\Psi \to \mu^+ + \mu^-$

- **Allowed**, neutral weak decay and EM decay both possible

g) / h)





1.4 Pion-proton scattering

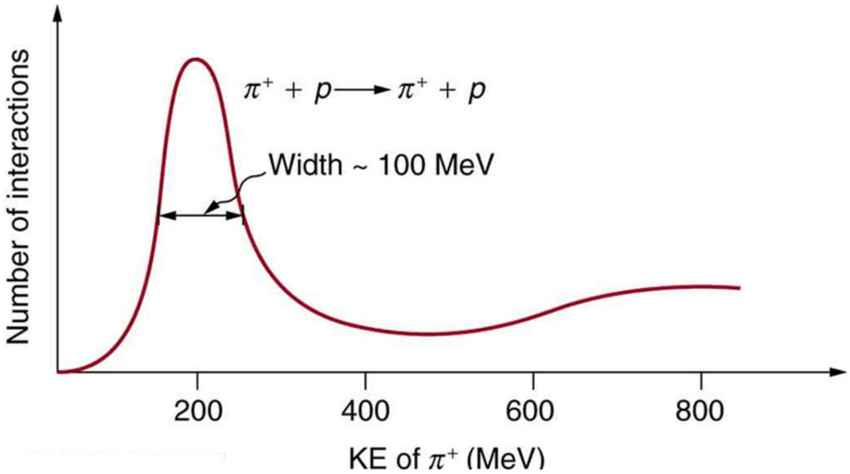


Figure 5: The probability of the interaction between a π^+ and a proton as a function of the incident kinetic π^+ energy.

Solutions

a) According to the description, this is a fixed-target experiment (with $p_p = 0$). This means that $E_p^2 = m_p^2c^4$, while $E_\pi^2 = m_\pi^2c^4 + p_{\pi^+}^2c^2$. We obtain the following invariant pion-proton mass:

$$\begin{aligned} m_{inv}c^2 &= \sqrt{(E_\pi + E_p)^2 - (p_\pi \cdot c)^2} \\ &= \sqrt{m_\pi^2c^4 + m_p^2c^4 + 2E_\pi m_p c^2} \\ &= \sqrt{m_\pi^2c^2 + m_p^2c^2 + 2(E_{kin,\pi} + m_\pi c^2)m_p c^2}. \end{aligned}$$

b) **Mass:** From the figure, we obtain $E_{kin,\pi} \approx 200\,\text{MeV}$. Using a), we calculate

$$\begin{aligned} m_X c^2 = m_{inv}c^2 &= \sqrt{m_\pi^2c^2 + m_p^2c^2 + 2(E_{kin,\pi} + m_\pi c^2)m_p c^2} \\ &\approx \sqrt{139.6^2 + 938.3^2 + 2 \cdot (200 + 139.6) \cdot 938.3}\ \text{MeV} \\ &\approx 1239.8\ \text{MeV} \end{aligned}$$

Charge: Due to conservation of charge, the particle should have a charge of 2.
(**Baryon number:** Since the proton is a baryon ($B = 1$), the pion is a meson ($B = 0$) and the baryon number is conserved, the created particle should be a baryon.)

$\rightarrow$We conclude that the created particle X is Δ^{++} ($m_\Delta = 1232$ MeV/c^2).

c) Strong interaction (+ some explanation).

d) Energy-time uncertainty relation: $\Delta E \cdot \Delta t \geq \hbar/2$.
To be entirely correct, one should use $\Gamma = 100$ MeV $= 2\Delta E$. The factor 2 was not treated explicitly in class, which is taken into account while grading.
This yields the following lifetime:

$$\tau = \frac{\hbar}{\Gamma} \approx 6.6 \cdot 10^{-24}\,\text{s}.$$

.