INSTITUUT VOOR THEORETISCHE FYSICA
UNIVERSITEIT UTRECHT

EXAM Field Theory in Particle Physics

Wednesday, June 11, 2014, 10.00 - 13.00, BBL205.

1) Start every exercise on a **separate** sheet.

2) Write on each sheet: your name and initials. In addition, write on the *first* sheet: your address, postal code and indicate whether you follow the master's programme in theoretical physics.

3) Please write legibly and clear.

4) The exam consists of **three** exercises.

1. Gauge theory of SU(N) with matter in the adjoint representation

Consider a non-abelian gauge theory based on the group SU(N) with anti-hermitian, traceless, generators t_a, coupled to N^2-1 scalar fields described by a hermitian, traceless matrix Φ, transforming in the adjoint representation of the gauge group. We write the gauge fields in Lie-algebra valued form, $W_\mu(x) = W_\mu{}^a(x)\, t_a$, with $\mathrm{Tr}[t_a\, t_b] = -\delta_{ab}$. Also the scalar fields can be decomposed in this way, $\Phi(x) = \mathrm{i}\varphi^a(x)\, t_a$. Hence the local SU($N$) transformation rules are

$$W_\mu \to U W_\mu U^{-1} + (\partial_\mu U)U^{-1}, \qquad \Phi \to U\Phi U^{-1}. \tag{1}$$

i) Give the expression for the field strength $G_{\mu\nu}$ and for the covariant derivative $D_\mu\Phi$ and specify (you don't need to derive it) how they transform under gauge transformations.

ii) Show that the following Lagrangian is gauge invariant,

$$\begin{aligned}\mathcal{L} = {}& \tfrac{1}{4g^2}\mathrm{Tr}\big[G_{\mu\nu}G^{\mu\nu}\big] \\ & -\tfrac{1}{2}\mathrm{Tr}\big[D_\mu\Phi\, D^\mu\Phi\big] - \tfrac{1}{2}\mu^2\mathrm{Tr}\big[\Phi^2\big] - \lambda_1\big(\mathrm{Tr}\big[\Phi^2\big]\big)^2 - \lambda_2\mathrm{Tr}\big[\Phi^4\big]\,.\end{aligned} \tag{2}$$

iii) We now add the following gauge-fixing term to the Lagrangian

$$\mathcal{L}_{\mathrm{g.f}} = -\tfrac{1}{2}(F^a)^2 \ \text{ with } \ F^a = \lambda\, n^\mu\, W_\mu{}^a\,, \tag{3}$$

where λ is an arbitrary constant, and n^μ denotes a constant vector. What is the gauge variation of F^a? From the result write down the associated ghost Lagrangian. Prove that the ghost loops will not depend on the parameter λ.

iv) In the limit $\lambda \to \infty$ the gauge field propagator $\Delta_{\mu\nu}(k)$ satisfies $n^\mu\,\Delta_{\mu\nu}(k) = 0$. Hence this limit corresponds to taking the axial gauge condition $n^\mu W_\mu{}^a = 0$. Explain now that the ghost loops cannot contribute to amplitudes with external gauge and matter fields in the $\lambda \to \infty$ limit.

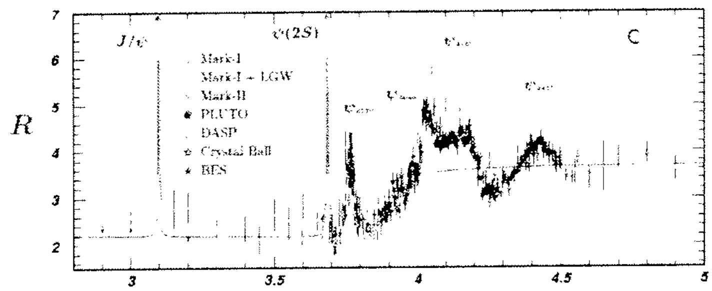


Figure 1: *The ratio R versus $\sqrt{s}$ [GeV]. The combined data from a number of experiments, including peaks of particle resonances.*

v) Consider including the following gauge invariant term to the Lagrangian

$$\mathcal{L}' = G\,\mathrm{Tr}\big[\,(D_\mu\Phi)\;(D_\nu\Phi)\;G_{\rho\sigma}\big]\varepsilon^{\mu\nu\rho\sigma}\,. \tag{4}$$

where G is a new coupling constant. This term contains three derivatives (one is implicit in $G_{\rho\sigma}$). Show, using the Bianchi identity for the gauge fields, that $\mathcal{L}'$ can be reduced to a term with only two derivatives.

2. Quantum Chromodynamics, the ratio R and IR divergences

R is the ratio of cross sections in e^+e^- collisions at center of mass energy $\sqrt{s}$,

$$R(s) = \frac{\sigma[e^+e^- \to \text{ hadrons}](s)}{\sigma[e^+e^- \to \mu^+\mu^-](s)}\,. \tag{1}$$

A measurement of this ratio is shown in Fig. 1. In this problem we consider perturbative QCD aspects of this quantity. QCD is an SU(3) gauge theory, with n_f quark flavours, each transforming as a triplet. Its Lagrangian reads

$$\mathcal{L}_{\mathrm{QCD}} = -\frac{1}{4}\mathrm{Tr}(G_{\mu\nu}G^{\mu\nu}) - \sum_{f=1}^{n_f}\bar{\psi}_f(\not{D} + m_f)\psi_f\,, \tag{2}$$

with

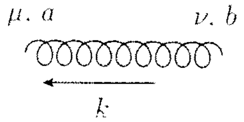
$$D_\mu = \partial_\mu - g_s W_\mu\,, \qquad G_{\mu\nu} = -g_s\,[D_\mu, D_\nu]\,, \tag{3}$$

and n_f the number of quark flavours. The gluons are written as $W_\mu = W_\mu{}^a t_a$ and the QCD coupling constant is g_s. In analogy to the fine structure constant in QED, we define

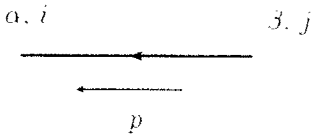
$$\alpha_s = \frac{g_s^2}{4\pi}\,. \tag{4}$$

and the quarks interact with photons accoring to the usual QED interaction $\mathcal{L}^{QED} = ie\,Q_f\,\bar{\psi}_f\gamma_\mu\psi_f$.

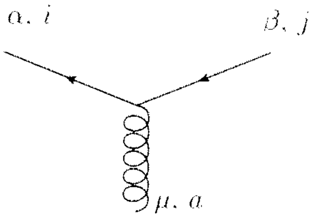
The relevant QCD Feynman rules are (we use Feynman gauge and take $\mathrm{Tr}[t_a\, t_b] = -\delta_{ab}$)



$$= \frac{1}{\mathrm{i}(2\pi)^4} \frac{\eta_{\mu\nu}\,\delta_{ab}}{k^2} \tag{5}$$



$$= \frac{\delta_{ij}}{\mathrm{i}(2\pi)^4} \left(\frac{1}{\mathrm{i}\not{p} + m} \right)_{\alpha\beta} \tag{6}$$



$$= \mathrm{i}(2\pi)^4 g_s\, (\gamma^\mu)_{\alpha\beta}\, (t_a)_{ij} \tag{7}$$

The lowest-order diagrams that contribute to the numerator and the denominator are given in Fig. 2.

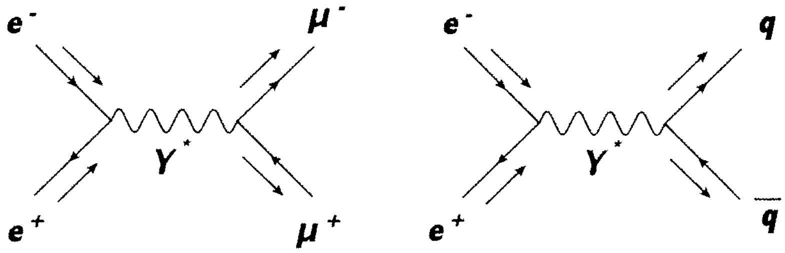


Figure 2: *The lowest order diagrams for numerator (rightmost) and denominator (leftmost) in the definition of R. The external particles are indicated, and γ^* indicates the intermediate off-shell photon. The quarks will all eventually become hadrons.*

i) We first neglect QCD interactions. Argue that we may express the ratio R as

$$R(s) = K \sum_f Q_f^2\, \theta(s - 4m_f^2)\,, \tag{8}$$

where K is a constant (in the approximation where we ignore differences in mass of the outgoing fermions), and where the quark flavour f has electric charge Q_f. How is K related to properties of SU(3)? Explain why R is approximately 3.3 in Fig. 1 for the largest value of $\sqrt{s}$ shown. [Realize that there are four different quark flavours with masses below $4\,\mathrm{GeV}c^{-2}$, two with $Q = -\frac{1}{3}$ and two with $Q = \frac{2}{3}$.]

In Fig. 3 we display one of the one-loop diagrams contributing to the numerator of the R ratio.

ii) Consider of this diagram only the part inside the box, and consider the quarks to be massless. Show that its expression, in n dimensions, reads

$$\Lambda^\mu_{ij}(p) = \mathrm{i}e\, Q_f\, g_s^2\, (t_a\, t^a)_{ij} \int \frac{d^n l}{\mathrm{i}(2\pi)^n}\, \frac{\bar{u}(p_1)\gamma_\alpha(-\mathrm{i}\not{p}_1 - \mathrm{i}\not{l})\gamma^\mu(\mathrm{i}\not{p}_2 - \mathrm{i}\not{l})\gamma^\alpha v(p_2)}{(l-p_2)^2\,(l+p_1)^2\, l^2}\,. \tag{9}$$

where $\bar{u}(p_1)$ and $v(p_2)$ are the on-shell spinors associated with the outgoing quark and anti-quark. Here we note that Λ^μ carries the the colour indices of the quark and anti-quark.

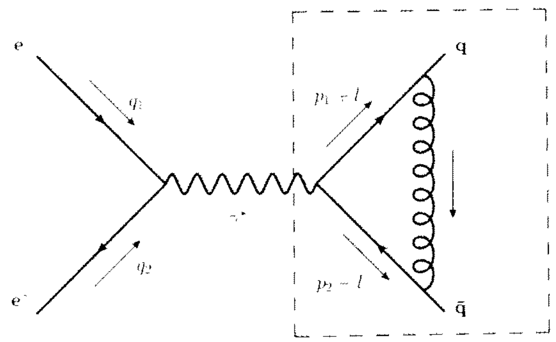


Figure 3: *One of the one-loop diagrams contributing to the numerator of the R ratio. The part in the box is the subject of question 2.ii).*

When one calculates this diagram in dimensional regularization, one finds among the terms the following one-loop scalar integral.

$$\int \frac{d^n l}{(2\pi)^n} \frac{1}{(l+p_1)^2\,(l-p_2)^2\,l^2} \,. \tag{10}$$

iii) Using Feynman parameters, show that this expression may be written as[1]

$$2\int \frac{d^n l}{(2\pi)^n} \int_0^1 dx \int_0^1 dy \frac{y}{[l^2 + 2xy\,l\cdot p_1 - 2(1-y)\,l\cdot p_2]^3} \,. \tag{11}$$

Show, writing $n = 4 + \varepsilon$, that this is equal to

$$\frac{i\pi^{2+\varepsilon/2}}{(2\pi)^{4+\varepsilon}} \Gamma(1-\varepsilon/2)(2p_1\cdot p_2)^{-1+\varepsilon/2} \int_0^1 dx\, x^{-1+\varepsilon/2} \int_0^1 dy\, y^{\varepsilon/2}(1-y)^{-1+\varepsilon/2} \tag{12}$$

iv) Explain how this leads to a double pole in ε. What are the physical origins of this double pole?

v) The full answer at order α_s in perturbative QCD, when including all diagrams, both virtual and real is finite and simply equal to

$$R(s) = K \sum_f Q_f^2\, \theta(s - 4m_f^2) \left(1 + \frac{\alpha_s}{\pi}\right) \,. \tag{13}$$

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[1]You may use the following results in your calculations.

$$\frac{1}{A^\alpha B^\beta} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 dx \frac{x^{\alpha-1}(1-x)^{\beta-1}}{[xA + (1-x)B]^{\alpha+\beta}} \,.$$

$$\int \frac{d^n q}{(2\pi)^n} \frac{1}{(q^2+m^2)^\alpha} = \frac{i\pi^{n/2}}{(2\pi)^n} \frac{\Gamma(\alpha - n/2)}{\Gamma(\alpha)} (m^2)^{n/2-\alpha} \,.$$

$$\int_0^1 dz\, z^{p-1}(1-z)^{q-1} = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \,.$$

where $\Gamma(z) = \Gamma(1+z)/z$ is the Euler gamma function. For small z one may approximate $\Gamma(1+z) \simeq 1 - \gamma_E z$.

Describe briefly how the infrared divergences such as we just found in fact cancel to reach this result.

3. The Higgs boson mass hierarchy problem

Here we study one of the most central problems in modern day particle physics: the self-energy corrections to the mass of the scalar Higgs boson h. This problem motivates many models beyond the Standard Model. We consider the following Lagrangian for h coupled to a fermion ψ with strength λ.

$$\mathcal{L}_{h\psi} = -\tfrac{1}{2}\left[(\partial_\mu h)^2 + m_h^2 h^2\right] - \bar{\psi}\not{\partial}\psi - m_\psi \bar{\psi}\psi + \lambda\, h\, \bar{\psi}\psi . \tag{1}$$

i) Consider the one-loop self-energy diagram for the scalar field h with a fermion loop, which is of order λ^2. Do you expect this diagram to be divergent, and if so what is its degree of divergence? Describe how renormalization would cancel the divergences. For simplicity we will assume that the mass correction can be evaluated by setting the external momentum in this diagram to zero. Argue that strictly speaking this is only correct when $m_h = 0$.

ii) Write down the expression for this diagram, remembering that self-energy diagrams acquire an additional factor of $[-\mathrm{i}(2\pi^n)]^{-1}$. Using dimensional regularization, show that

$$\begin{aligned}\delta m_h^2 = &-\frac{\lambda^2 m_\psi^2 \mu^\epsilon}{4\pi^2}\left(\frac{1}{\varepsilon} + \frac{\gamma_E - 1}{2} + \frac{1}{2}\ln\left(\frac{m_\psi^2}{4\pi\,\mu^2}\right)\right)\\ &-\frac{\lambda^2 m_\psi^2 \mu^\epsilon}{2\pi^2}\left(\frac{1}{\varepsilon} + \frac{\gamma_E}{2} + \frac{1}{2}\ln\left(\frac{m_\psi^2}{4\pi\,\mu^2}\right)\right) .\end{aligned} \tag{2}$$

Next, we consider h additionally coupled to a complex scalar field φ, described by the Lagrangian

$$\mathcal{L}_{h\phi} = -\left[|\partial_\mu \phi|^2 + m_\phi^2\, |\phi|^2\right] - \kappa\, h^2\, |\varphi|^2 + v\,\kappa\, h\, |\phi|^2 , \tag{3}$$

with two coupling constants κ and $v\,\kappa$.

iii) Draw the extra one-loop diagrams that contribute to δm_h^2 and write down the corresponding expressions (without using a regulator). Assume again that the external momentum can be set to zero, and argue (without a new n-dimensional calculation!) that they give the same expression as (2), with m_ψ replaced by m_ϕ, and, for the first line in (2), λ^2 replaced by $-\frac{1}{2}\kappa$ and, for the second line, $4\,\lambda^2 m_\psi^2$ replaced by $-v^2\kappa^2$.

iv) Clearly, using $v = \frac{m_\psi}{\lambda}$, $m_\phi = m_\psi$ and $\kappa = 2\,\lambda^2$, the leading singularities cancel. But if $m_\phi^2 = m_\psi^2 + \delta^2$, what is the total correction to the mass of h? Assuming that m_ψ is very large ($m_\psi \gg 1\,\mathrm{GeV}c^{-2}$), can the corrections to the mass of h be made small? What do you learn from this?

v) **Bonus question**: Is this theory strictly renormalizable, and why (not)?