INSTITUUT VOOR THEORETISCHE FYSICA
UNIVERSITEIT UTRECHT

EXAM Field Theory in Particle Physics

Wednesday, July 2, 2014, 10.00 - 13.00, BBG 083.

1) Start every exercise on a **separate** sheet.

2) Write on each sheet: your name and initials. In addition, write on the *first* sheet: your address, postal code and indicate whether you follow the master's programme in theoretical physics.

3) Please write legibly and clear.

4) The exam consists of **three** exercises.

1. Renormalization in Quantum Chromodynamics and asymptotic freedom

Here we examine the notion of asymptotic freedom in QCD. The QCD Lagrangian is:

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{4}\text{Tr}(G_{\mu\nu}G^{\mu\nu}) - \sum_{f=1}^{n_f} \overline{\psi_f}(\not{D} + m_f)\psi_f \tag{1}$$

with

$$D_\mu = \partial_\mu - g_{s,0}A_\mu, \qquad [D_\mu, D_\nu] = -g_{s,0}G_{\mu\nu} \tag{2}$$

where $g_{s,0}$ is the unrenormalized ("bare") QCD coupling and n_f the number of quark flavors. The quark fields ψ_f transform in the fundamental representation of SU(3). The Lagrangian in (1) is invariant under local SU(3) transformations.

i) From the transformation rule for the covariant derivative, derive how the Lie-algebra valued field A_μ must transform under a finite SU(3) transformation. How does then the Lie-algebra valued field strength tensor $G_{\mu\nu}$ transform?

We could possibly add to the Lagrangian in (1) the term

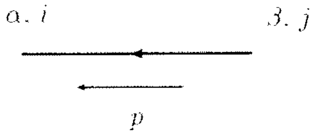
$$\mathcal{L}_\theta = \theta\,\text{Tr}(G_{\mu\nu}G_{\rho\sigma})\varepsilon^{\mu\nu\rho\sigma} \tag{3}$$

with $\varepsilon^{\mu\nu\rho\sigma}$ the 4-dimensional Levi-Civita symbol, which is fully anti-symmetric in all four of its indices.

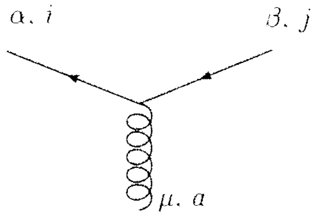
ii) [**Bonus question**] Argue that this term is odd under a parity transformation, under which for any vector (field) the space components get a minus sign, but the time component does not. Show that $\mathcal{L}_\theta$ is a total derivative.

We shall however *not* include this term and continue to work with the Lagrangian in (1).

Some of the QCD Feynman rules are



$$= \frac{1}{i(2\pi)^4}\left(\frac{1}{i\not{p}+m}\right)_{\alpha\beta}\delta_{ij} \tag{4}$$



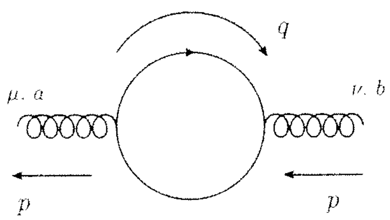
$$= i(2\pi)^4(-ig_s)(\gamma^\mu)_{\alpha\beta}(t_a)_{ij} \tag{5}$$

We define

$$\alpha_s = \frac{g_s^2}{4\pi}\,. \tag{6}$$

iii) First we consider the gluon self-energy diagram.

$$\Pi^{ab}_{\mu\nu}(p) \; = \tag{7}$$



Show that the expression for $\Pi^{ab}_{\mu\nu}(p)$ equals, in n dimensions,

$$\Pi^{ab}_{\mu\nu}(p) = g_s^2\sum_f\int\frac{d^nq}{i(2\pi)^n}\frac{\mathrm{Tr}\left(\gamma_\nu(-i(\not{p}+\not{q})+m_f)\gamma_\mu(-i\not{q}+m_f)\right)}{\left((p+q)^2+m_f^2\right)\left(q^2+m_f^2\right)}Tr(t_at_b)\,. \tag{8}$$

Argue that it must be proportional to $p_\mu p_\nu - \eta_{\mu\nu}p^2$. Give its superficial degree of divergence.

In dimension regularization ($n = 4+\epsilon$), the QCD coupling is renormalized as follows.

$$g_{s,0} = \mu^{-\epsilon/2}Z_g\,g_{s,R}(\mu) \tag{9}$$

$$Z_g = 1+\frac{1}{\epsilon}\alpha_{s,R}(\mu)\,\frac{\beta_0}{4\pi}+\ldots\,, \qquad \beta_0 = \frac{11}{3}C_A-\frac{2}{3}n_f \tag{10}$$

with μ the renormalization scale, $g_{s,R}(\mu)$ the renormalized coupling evaluated at scale μ and C_A the Casimir invariant for the adjoint representation.

iv) Discuss how the loop diagram in Eq. (8) contributes to the renormalization in Eq. (9). Draw the one-loop correction to the 3-gluon vertex involving a fermion loop and argue that it contributes to the renormalization of the QCD coupling constant on the basis of power counting.

v) Give the definition of the beta function for the coupling α_s. From the renormalization equations Eq. (9), Eq. (10), compute this beta function up to order $\alpha_{s,R}^2$ and obtain a differential equation govering the "running" of the QCD gauge coupling. Finally, show that the solution to this equation reads

$$\alpha_{s,R}(\mu) = \frac{4\pi/\beta_0}{\ln\left(\frac{\mu^2}{\Lambda^2}\right)} \tag{11}$$

for some scale Λ. This behavior of the coupling exhibits so-called asymptotic freedom. Explain what this means physically.

We now assume that we have computed a certain cross-section σ using QCD Feynman rules. The lowest order calculation is already of order α_s^2, and we assume that we have computed the full α_s^3 term for this cross-section,

$$\sigma(Q^2) = \alpha_{s,0}^2\left(Q^{2\epsilon}F(Q^2)\right) + \alpha_{s,0}^3\left(\frac{-1}{\epsilon}\frac{\beta_0}{\pi}Q^{3\epsilon}F(Q^2) + G(Q^2)\right) \tag{12}$$

with F, G finite functions, and Q a large energy scale typical for that cross-section. The $\mathcal{O}(\alpha_s^3)$ correction term contains a UV divergence.

vi) Show that this divergence can be removed by renormalizing the coupling and that one finds, to order $\alpha_{s,R}^3$, after renormalization,

$$\sigma(Q^2) \quad = \quad \alpha_{s,R}(\mu)^2 F(Q^2) + \alpha_{s,R}(\mu)^3\left(-\ln\left(\frac{Q}{\mu}\right)\frac{\beta_0}{\pi}F(Q^2) + G(Q^2)\right). \tag{13}$$

2. Gravitons

Like the Standard Model, gravitation is also based on a non-abelian gauge group, although this gauge group (consisting of general coordinate transformations) is infinite-dimensional and of a different type than the gauge groups that one encounters in particle physics. Nevertheless, certain features are very similar. The gauge field of gravity is a symmetric rank-two field $h_{\mu\nu}(x)$ which describes the gravitational degrees of freedom. The particles associated with the gravitational field are massless spin-2 particles, called *gravitons*.

Here we consider free massless gravitons in four space-time dimensions. Obviously the free field must (at least) satisfy the field equation $\Box h_{\mu\nu} = 0$.

i) How many field components does $h_{\mu\nu}$ comprise?

Just as for the photon, a Lorentz covariant description will require gauge invariance to reduce the number of degrees of freedom. For the graviton, the (linearized) gauge transformations take the form $\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$, where the $\xi_\mu(x)$ are four arbitrary functions of the space-time coordinates. Because of this gauge invariance the gravitons must couple to a conserved tensor source $T^{\mu\nu}$, symmetric in μ, ν and subject to $\partial_\mu T^{\mu\nu} = 0$.

ii) In light of this gauge invariance, how many off-shell degrees of freedom should $h_{\mu\nu}$ comprise?

iii) Let us consider the coupling of $h_{\mu\nu}$ to photons. Argue that, because of *electromagnetic* gauge invariance and parity reversal symmetry, the expression for $T^{\mu\nu}$ can be written in terms of the photon field strength as $T^{\mu\nu} = \alpha\, \eta^{\mu\nu}\, {F_{\rho\sigma}}^2 + \beta\, F^{\mu\rho}\, F^{\nu}{}_{\rho}$, up to terms with more than two derivatives. Derive now a linear relation between the two parameters α and β by imposing the condition $\partial_\mu T^{\mu\nu} = 0$ and by making use of the QED field equation (in the absence of matter) and the Bianchi identity.

We will now consider the exchange of a graviton between two arbitrary conserved sources, $T^{\mu\nu}$ and $T^{\mu\nu\prime}$, not necessarily associated with the photon field strength. This is the analog of what was done in class where we considered the exchange of a (virtual) photon between two conserved currents $\partial_\mu J^\mu = \partial_\mu J^{\mu\prime} = 0$, where we found

$$\bar{J}^\mu(k)\, \Delta_{\mu\nu}(k)\, J'^\nu(k) = \frac{1}{\mathrm{i}(2\pi)^4} \left\{ \frac{\bar{\boldsymbol{J}}_\perp(k)\cdot \boldsymbol{J}'_\perp(k)}{k^2} - \frac{\bar{J}_0(k)\, J'_0(k)}{|\boldsymbol{k}|^2} \right\} . \tag{1}$$

Observe that we have eliminated the longitudinal component of $\boldsymbol{J}$ (i.e. the component parallel to the momentum $\boldsymbol{k}$) by using current conservation. Note also that we have used that the propagator residue is equal to $\eta_{\mu\nu}$, up to terms proportional to k_μ or/and k_ν which vanish for conserved sources.

iv) Describe the physical consequences of the result (1) for the number of physical photon polarizations.

v) For simplicity let us assume that the propagator three-momentum is directed along the 3-axis, so that $k^i = 0$, where $i = 1, 2$ denote the transverse directions. Let us now decompose the source $T^{\mu\nu}(k)$ into $T^{33}(k)$, $T^{3i}(k)$, $T^{03}(k)$, $T^{00}(k)$, $T^{0i}(k)$ and $T^{ij}(k)$, where k^μ denotes the incoming (or outgoing) *off-shell* momentum. Show that the first three components can be decomposed in terms of the last three. How many independent components do T^{ij}, T^{00} and T^{0i} have? How does this counting relate to your answer in question ii)?

Consider the analog of (1) for the exchange of a graviton, where the propagator carries two symmetrized index pairs, $(\mu\nu)$ and $(\rho\sigma)$. Hence we consider the expression $\bar{T}^{\mu\nu}(k)\, \Delta_{\mu\nu,\rho\sigma}(k)\, T^{\rho\sigma\prime}(k)$. As suggested by the previous case we assume that the propagator $\Delta_{\mu\nu,\rho\sigma}(k)$ has an overall factor $[\mathrm{i}(2\pi^4)\, k^2]^{-1}$ and a residue factor that consists of a linear combination of $\frac{1}{2}(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho})$ and an optional term $\gamma\, \eta_{\mu\nu}\eta_{\rho\sigma}$, where γ is a parameter.

vi) To find the expression for $\bar{T}\,\Delta\, T'$ consider first the residue terms, $\bar{T}^{\mu\nu}\, T_{\mu\nu}'$ and $\gamma\, \bar{T}^{\mu}{}_{\mu}\, T^{\rho}{}_{\rho}{}'$, and express them in terms of the components T^{00}, T^{0i} and T^{ij} for both tensor sources.

vii) Write the various contributions similar to the first term in (1) and determine the number of physical gravitons for an arbitrary value of γ. Determine the value of γ for which the generic number of graviton states decreases by one. Can you explain this phenomenon?

3. Combinatorics of vertices

Consider the Lagrangian for a multi-component scalar field $\vec{\phi} = (\phi^1, \phi^2, \ldots, \phi^N)$ equal to

$$\mathcal{L} = -\tfrac{1}{2}\partial_\mu \vec{\phi} \cdot \partial^\mu \vec{\phi} - \tfrac{1}{2}M^2\, \vec{\phi} \cdot \vec{\phi} - \lambda\, (\vec{\phi} \cdot \vec{\phi})^2 \,. \tag{1}$$

When dealing with external lines with indices, one can include all line attachments into the definition of the vertex and divide by a factor of $n!$, where n is the number of lines emanating from the vertex. Here we have only a single four-point vertex proportional to the coupling λ.

i) Show that, according to the above prescription, this vertex can be written as

$$V^{abcd} = -\tfrac{1}{3}\lambda \big[\delta^{ab}\delta^{cd} + \delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc}\big] \,, \qquad a, b, c, d = 1, \ldots, N \,, \tag{2}$$

where we suppressed the standard factor $\mathrm{i}(2\pi)^4$ times the momentum-conserving delta function. Where does the final numerical factor of $1/3$ comes from?

ii) When considering ϕ-ϕ-scattering, the explicit expressions for the diagrams will also involve the square of the vertex (2). Show that this square is

$$V^{abef}V^{efcd} = \tfrac{1}{9}\lambda^2\big[(N+4)\delta^{ab}\delta^{cd} + 2\delta^{ac}\delta^{bd} + 2\delta^{ad}\delta^{bc}\big] \tag{3}$$

iii) Now write down the one-loop diagrams for ϕ-ϕ-scattering with the external momenta p^a, p^b, p^c, p^d taken as incoming. Argue that there are three such diagrams, each involving the function

$$I((p_1+p_2)^2, M^2) = \int \mathrm{d}^4 q \, \frac{1}{(q^2+M^2)\big((q+p_1+p_2)^2+M^2\big)} \,, \tag{4}$$

depending on the three independent sums of two of the external momenta, i.e. p^a+p^b, p^a+p^c and p^a+p^d.

iv) Consider the case of $N=1$ and compute the combinatorial factor for any of the three diagrams.