Institute for Theoretical Physics
Utrecht University

EXAM Field Theory in Particle Physics

Wednesday, April 15, 2015, 10.30 - 12.30, BBG 083.

1) Start every exercise on a **separate** sheet.
2) Write on each sheet: your full name and student number. In addition, write on the *first* sheet: your address, and indicate whether you follow the master's programme in theoretical physics.
3) Please write legibly and clear. *Keep your answers brief and to the point!*
4) The exam consists of **two** exercises.

1. Gauge theory of SU(N) with fermions and scalars

Consider a non-abelian gauge theory based on the group SU(N) with anti-hermitian, traceless, generators t_a, coupled to N fermion fields, and also to $N^2 - 1$ scalar fields. We describe the fermions through an N-component vector ψ, and the scalars by a hermitian, traceless matrix Φ, transforming in the adjoint representation of the gauge group. We write the gauge fields in Lie-algebra valued form, $W_\mu(x) = W_\mu{}^a(x)\, t_a$, with $\mathrm{Tr}[t_a\, t_b] = -\delta_{ab}$. Also the scalar fields can be decomposed in this way, $\Phi(x) = \mathrm{i}\phi^a(x)\, t_a$. Hence the local SU($N$) transformation rules are

$$W_\mu \to U W_\mu U^{-1} + (\partial_\mu U)U^{-1}, \qquad \psi \to U\psi, \qquad \Phi \to U\Phi U^{-1}\,. \tag{1}$$

?
i) Give the expression for the field strength $G_{\mu\nu}$, for the covariant derivative $D_\mu\psi$, and for the covariant derivative $D_\mu\Phi$ and specify (you don't need to derive it) how they transform under gauge transformations.

✓
ii) Show that the following Lagrangian is gauge invariant,

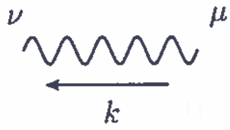
$$\begin{aligned}\mathcal{L} = {}& \tfrac{1}{4g^2}\mathrm{Tr}\big[G_{\mu\nu}G^{\mu\nu}\big] - \overline{\psi}\not{D}\psi \\ & - \tfrac{1}{2}\mathrm{Tr}\big[D_\mu\Phi\, D^\mu\Phi\big] - \tfrac{1}{2}\mu^2\mathrm{Tr}\big[\Phi^2\big] - \lambda_1\big(\mathrm{Tr}\big[\Phi^2\big]\big)^2 - \lambda_2\mathrm{Tr}\big[\Phi^4\big]\,.\end{aligned} \tag{2}$$

✓
iii) Notice that in (2) there are no terms yet that couple the ψ field to the Φ field. Argue that there is only *one* gauge-invariant renormalizable interaction that couples 2 ψ fields to 1 Φ field. Argue that there are *two* dimension-5 interactions that couple 2 ψ fields to 2 Φ fields.

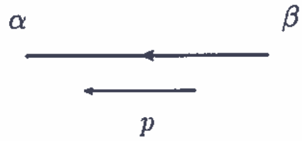
2. A one-loop diagram in QED and renormalization

In this problem we examine ultraviolet divergences in a one-loop diagram in quantum-electrodynamics.

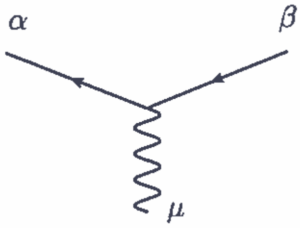
We recall the Feynman rules



$$= \frac{1}{i(2\pi)^4}\frac{1}{k^2}\Big(\eta_{\mu\nu} - (1-\frac{1}{\lambda^2})\frac{k_\mu k_\nu}{k^2}\Big)$$



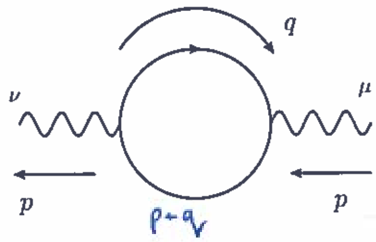
$$= \frac{1}{i(2\pi)^4}\left(\frac{1}{i\not{p}+m}\right)_{\alpha\beta}$$



$$= i(2\pi)^4(-ie)(\gamma^\mu)_{\alpha\beta} \tag{1}$$

We consider the photon self-energy diagram

$$\Pi_{\mu\nu}(p) =$$



(2)

i) Show that the expression for $\Pi_{\mu\nu}(p)$, in n dimensions, equals,

✓

$$\Pi_{\mu\nu}(p) = \frac{ie^2}{(2\pi)^n}\int d^nq\, \frac{\mathrm{Tr}\left(\gamma_\nu(-i(\not{p}+\not{q})+m)\gamma_\mu(-i\not{q}+m)\right)}{((p+q)^2+m^2)\,(q^2+m^2)}. \tag{3}$$

where a factor $-i(2\pi)^n$ has been extracted from the diagram.

?

ii) Argue that $\Pi_{\mu\nu}(p)$ is at most quadratically divergent in four space-time dimensions.

iii) Argue that gauge invariance implies the Ward identity $p^\mu\Pi_{\mu\nu}(p) = 0$, and that we can therefore write

?

$$\Pi_{\mu\nu}(p) = (\eta_{\mu\nu}p^2 - p_\mu p_\nu)\,\Pi(p^2)\,. \tag{4}$$

(We may assume that $\Pi(0)$ is equal to a constant.) What does the form in (4) imply for the photon mass in the one-loop approximation? Do you expect that the degree of divergence of the function $\Pi(p^2)$ is also quadratic?

iv) By contracting $\Pi_{\mu\nu}(p)$ with $\eta^{\mu\nu}$ (in n dimensions) one can derive an expression for $\Pi(p^2)$ in terms of an n-dimensional momentum integral. Derive the corresponding expression and decompose it in terms of two integrals,

almost

$$I_0(m^2) = \frac{1}{(2\pi)^n}\int\frac{d^nq}{q^2+m^2}\,,$$

$$I(p^2,m^2,m^2) = \frac{1}{(2\pi)^n}\int\frac{d^nq}{[(q-p/2)^2+m^2]\,[((q+p/2)^2+m^2]}\,. \tag{5}$$

Remark: recall that the trace over the identity (by convention) gives a factor of 4 in n dimensions.

v) **Bonus question** *Prove in dimensional regularization that*

$$\int \frac{d^n q}{q^2 + m^2} = \frac{2\,m^2}{2-n} \int \frac{d^n q}{(q^2+m^2)^2} \,. \tag{6}$$

You may use that $n \int d^n q f(q) = -\int d^n q\, q^\mu \frac{\partial f(q)}{\partial q^\mu}$.

vi) Use (6) to write the result for $\Pi(p^2)$ exclusively in terms of the function $I(p^2, m^2, m^2)$, to obtain the form,

$$\Pi(p^2) = 4ie^2 \Big\{ -\frac{1}{2}\left(\frac{n-2}{n-1}\right) I(p^2, m^2, m^2) + \frac{2m^2}{n-1} \frac{I(p^2, m^2, m^2) - I(0, m^2, m^2)}{p^2} \Big\} . \tag{7}$$

Verify that the degree of divergence of this result agrees with your original prediction in iii).

vii) Argue that the second term in (7) is ultraviolet finite, and that the divergence must be given by

$$\Pi^{\text{infinite}}_{\mu\nu}(p) = -2ie^2 (\eta_{\mu\nu} p^2 - p_\mu p_\nu) \left(\frac{n-2}{n-1}\right) I(0, m^2, m^2) \,. \tag{8}$$

viii) Using

$$\int \frac{d^n q}{(2\pi)^n} \frac{1}{(q^2+m^2)^\alpha} = \frac{i\pi^{n/2}}{(2\pi)^n} \frac{\Gamma(\alpha - n/2)}{\Gamma(\alpha)} (m^2)^{n/2-\alpha} \,, \tag{9}$$

where $\Gamma(z) = \Gamma(1+z)/z$ is the Euler gamma function, ($\Gamma(z) = \frac{1}{z} - \gamma_{\rm E} + \mathcal{O}(z)$) show that (for $n = 4 + \epsilon$)

$$\Pi_{\mu\nu}(p^2) = -\Big[\frac{e^2}{6\pi^2}\frac{1}{\epsilon} + \text{finite terms}\Big] (\eta_{\mu\nu} p^2 - p_\mu p_\nu) \,. \tag{10}$$

ix) Indicate how a renormalization of the A_μ field can provide a counterterm that cancels this divergence.