Institute for Theoretical Physics
Utrecht University

Midterm EXAM Field Theory in Particle Physics

Wednesday, April 10, 2019, 10.00 - 12.00, BBG 023 and BBG 001.

1) Start each exercise on a **separate** sheet.
2) Write on each sheet: your full name and student number.
3) Please write legibly and clear. *Keep your answers brief and to the point!*
4) The exam consists of two exercises.

1. Two kinds of QCD

We consider some symmetry properties of QCD, as a non-abelian gauge theory based on SU(3), and a variant of it, which we call QCD2.

Consider a fermion field ψ with Lagrangian

$$\mathcal{L}_\psi = -\bar{\psi}\not{D}\psi - m\bar{\psi}\psi\,, \qquad D_\mu = \partial_\mu - gA_\mu\,. \tag{1}$$

The field ψ transforms as

$$\psi^i(x) \to \psi'^i(x) = U^i{}_j(x)\,\psi^j(x)\,, \tag{2}$$

where $U(x) = \exp(g\,\xi^a(x)t_a)$, with generators t_a in the fundamental, or defining, representation of SU(3). They obey the Lie-algebra relation

$$[t_a, t_b] = f_{ab}{}^c\, t_c\,, \tag{3}$$

with $f_{ab}{}^c$ the structure constants. The generators in the fundamental representation are normalized as

$$\mathrm{Tr}\big[t_a t_b\big] = -\frac{1}{2}\delta_{ab}\,. \tag{4}$$

a) By requiring that QCD Lagrangian (1) is invariant under the transformation in (2), derive the transformation rule of the gauge field A^a_μ.

b) Given the covariant derivative in (1), work out the field-strength tensor using

$$[D_\mu, D_\nu] = -gG_{\mu\nu}\,. \tag{5}$$

and show that it transforms in the adjoint representation. You do not need to make the gauge indices explicit.

Next we consider a Lagrangian similar to (1),

$$\mathcal{L}_\psi = -\bar{\psi}_{ij}(\partial\!\!\!/ + m)\,\mathrm{I}^{ij}{}_{kl}\,\psi^{kl} + g\bar{\psi}_{ij}(A\!\!\!/)^{ij}{}_{kl}\psi^{kl}\,, \tag{6}$$

but now ψ transforms as

$$\psi^{ij}(x) \to \psi'^{ij}(x) = U^i{}_k(x)\,U^j{}_l(x)\,\psi^{kl}(x)\,. \tag{7}$$

with ψ in the symmetric represention ($\psi^{ij} = \psi^{ji}$). Here $\mathrm{I}^{ij}{}_{kl}$ is the identity matrix for the six independent elements of ψ^{ij}. The matrix U is again in the fundamental representation of the gauge group.

Let us call this theory QCD2.

c) From the Lagrangian in (6), derive first the inverse propagator $\Delta^{ij}{}_{kl}(k)$ in momentum space. Subsequently invert your result to give the propagator.

d) Show that the generators of this symmetric representation are given by

$$(t_a^{\mathrm{S}})^{ij}{}_{kl} = \frac{1}{2}\Big(\delta^i{}_k\,(t_a)^j{}_l + \delta^j{}_k\,(t_a)^i{}_l + \delta^i{}_l\,(t_a)^j{}_k + \delta^j{}_l\,(t_a)^i{}_k\Big)\,. \tag{8}$$

What do you expect for the commutator of two such generators? Construct the QCD2 covariant derivative when acting on ψ.

e) Consider now the vacuum polarization diagram $\Pi^{ab}_{\mu\nu}(k)$ in fig. 1. Argue that the $\Pi^{ab}_{\mu\nu}$ functions of QCD and QCD2 are proportional, and give an expression for the proportionality factor. *Note: you do not need to give the expression for the full diagram.*

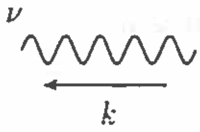
Figure 1: *Vacuum polarization diagram, where the external gauge bosons are gluons with adjoint indices a and b.*

2. Meson decays to two photons

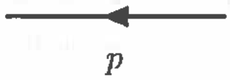
We consider a complex scalar field ϕ, representing π^+ and π^- particles, coupled to photons, with Lagrangian

$$\begin{aligned}\mathcal{L}_\phi = & -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi \\ & - \mathrm{i}eA_\mu[\phi^*(\partial^\mu\phi) - (\partial^\mu\phi^*)\phi] - e^2A_\mu^2\,\phi^*\phi\,. \end{aligned} \tag{1}$$

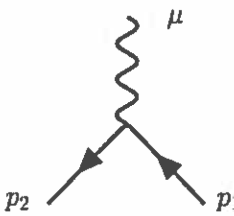
where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The Feynman rules corresponding to $\mathcal{L}_\phi$ read



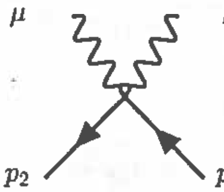
$$= \frac{1}{\mathrm{i}(2\pi)^4}\frac{1}{k^2}\left(\eta_{\mu\nu} - \left(1 - \frac{1}{\lambda^2}\right)\frac{k_\mu k_\nu}{k^2}\right)$$



$$= \frac{1}{\mathrm{i}(2\pi)^4}\frac{1}{p^2+m^2}$$



$$= \mathrm{i}(2\pi)^4(-\mathrm{i}e)(\mathrm{i}p_1^\mu + \mathrm{i}p_2^\mu)$$



$$= \mathrm{i}(2\pi)^4(-e^2)(\eta^{\mu\nu})$$

To the Lagrangian in (1) we add a term for the interaction of a neutral rho-meson and charged pions (the rho-meson is a massive vector meson)·

$$\mathcal{L}_{\rho\pi\pi} = g\rho_\mu(\phi^*\partial^\mu\phi - (\partial^\mu\phi^*)\phi)\,. \tag{2}$$

a) There are three one-loop diagrams mediated by virtual pions that contribute to the decay $\rho \to \gamma(p) + \gamma(q)$. Two of these are shown in fig. 2, draw the remaining diagram.

b) Show that the diagram in fig. 2a corresponds to the expression (in $n = 4 + \varepsilon$ dimensions)

$$\mathcal{M}_{\mu\nu\lambda} = \frac{2\,ge^2}{\mathrm{i}(2\pi)^n}\int \mathrm{d}^n k\,\frac{\eta_{\mu\nu}(2k+p-q)_\lambda}{((k+p)^2+m^2)((k-q)^2+m^2)}\,, \tag{3}$$

where an overall factor of $\mathrm{i}(2\pi)^n$ has been extracted. What is the superficial degree of divergence of the diagram?

c) Show that the contribution of diagram 2a to the physical amplitude for the physical decay of a rho-meson into two photons - which involves the polarization vectors of the external vector particles- vanishes.

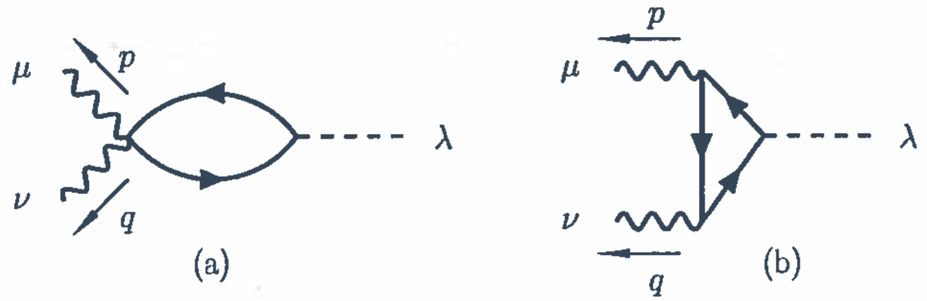


Figure 2: *Two of the diagrams contributing to the decay amplitude for $\rho \to \gamma\gamma$, where the vector indices μ, ν are those of the photons, and λ of the rho-meson.*

d) The vanishing of this diagram is due to the vector nature of the rho-meson. Show that if ρ had been a scalar boson the answer in $n = 4 + \varepsilon$ dimensions would have been proportional to

$$2\eta_{\mu\nu}\frac{\pi^{n/2}}{(2\pi)^n}\frac{2}{\varepsilon}\Gamma(1-\varepsilon/2)\, m^{\varepsilon}\int_0^1 \mathrm{d}x\left[1-\frac{m_\rho^2}{m^2}x(1-x)\right]^{\varepsilon/2}. \tag{4}$$

To show this you may use the relations

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{[xA+(1-x)B]^2} \tag{5}$$

$$\int \frac{\mathrm{d}^n q}{(2\pi)^n}\frac{1}{(q^2+m^2)^\alpha} = \frac{\mathrm{i}\pi^{n/2}}{(2\pi)^n}\frac{\Gamma(\alpha-n/2)}{\Gamma(\alpha)}(m^2)^{n/2-\alpha}, \tag{6}$$

where $\Gamma(z) = \Gamma(1+z)/z$ is the Euler gamma function.

e) Show that the diagram in fig. 2b corresponds to the expression (in $n = 4 + \varepsilon$ dimensions)

$$\mathcal{M}_{\mu\nu\lambda} = \frac{2\,ge^2}{\mathrm{i}(2\pi)^n}\int \mathrm{d}^n k \frac{(2\,k+p)_\mu\,(2\,k-q)_\nu(2k+p-q)_\lambda}{((k+p)^2+m^2)((k-q)^2+m^2)(k^2+m^2)}, \tag{7}$$

where again an overall factor of $\mathrm{i}(2\pi)^n$ has been extracted. What is the superficial degree of divergence of this diagram?

f) **Bonus:** Show that for the physical decay amplitude of $\rho \to \gamma\gamma$ the 3rd diagram mentioned in question a) cancels the result of diagram 2b.

In fact, L. Landau and C.N. Yang have shown on quite general grounds that a massive vector boson cannot decay to two photons.