Answers to the exam ISP of December 19, 2005

1. Counting time in hours:

   a. Taking $t = 2$ in $P(N(t) = 20) = e^{-10t}(10t)^{20}/20!$ yields 0.089

   b. $P(N(0.01) \geq 1) = 1 - e^{-0.1}$, or $P(X \leq 0.01)$ where X is an interarrival time, hence exponentially distributed with parameter 10.

   c. For the remaining interarrival time X we have $EX = 1/10$ hours, so 6 minutes *after 12.00 hours*. Hence at 12.06 hours.

   d. Let $p = P(\text{job has } > 3 \text{ pages}) = 1 - e^{-2}(1+2+2+4/3) \approx 0.143$. Then $\{M(t)\}$ is a Poisson process with rate $10p$, so $P(M(t) = m) = e^{-10pt}(10pt)^m/m!$.

2. a. Given the present, say $X_n = i$, the process will jump to 0 or $i+1$ with probabilities that do not depend on $X_{n-1}, X_{n-2}, \ldots$. Hence the process is a DTMC. It is irreducible (the path 0, 1, 2, 3, 4, 5, 6, 0 has positive probability so all states communicate), aperiodic (GCD(2,3,4,...)=1, so period of state 0, and hence all other states, is 1), and not transient but recurrent (finite closed class).

   b. Solving $\pi = \pi P$ (where $P_{i,0} = i/6$, $P_{i,i+1} = 1 - i/6$, other entries equal 0) yields $\pi = \pi_0(1, 1, 5/6, 20/36, 60/6^3, 120/6^4, 120/6^5)$, so that $\sum \pi_i = 1$ yields $\pi_0 = (1 + 1 + 5/6 + 20/36 + 60/6^3 + 120/6^4 + 120/6^5)^{-1} = 324/1223 \approx 0.265$.

   c. $\lim_{n\to\infty} P(X_{n-1} = 2, X_n = 0) = \lim_{n\to\infty} P(X_{n-1} = 2)P(X_n = 0|X_{n-1} = 2) = \pi_2 P_{2,0} = 5/6\pi_0 1/3 = 324/1223$.

   d. $m_0 = 1/\pi_0 = 1223/324 \approx 3.77$.

3. Counting time in hours:

   a. Suppose $X(t) = n$. Then time until next arrival (departure) has exponential distribution with parameter $\lambda = 30$ ($n\mu = 60n$). Hence the minimum of these also has exponential distribution, with parameter $\lambda + n\mu = 30 + 60n$

   b. Solving balance equations: $\lambda\pi_{n-1} = n\mu\pi_n$, so $\pi_n = \pi_{n-1}\lambda/(n\mu) = \ldots = \pi_0(\lambda/\mu)^n/n!$, where $\pi_0 = (\sum(1/2)^n/n!)^{-1} = e^{-1/2}$. Hence $\pi_n = e^{-1/2}(1/2)^n/n!$.

   c. Since $\{X(t), t \geq 0\}$ is a birth-death process, it is time-reversible and we can consider the process $\{\tilde{X}(t)\}$ on the truncated state space $\{0, 1, \ldots, 4\}$. For n in this set we find $\tilde{\pi}_n = \pi_n/(\sum_{i=0}^{4} \pi_i)$, so in particular $\tilde{\pi}_4 = \frac{(1/2)^4/4!}{1+1/2+(1/2)^2/2!+(1/2)^3/3!+(1/2)^4/4!} = 1/633 \approx 0.0016$.

4. a. Yes, let X_n be the time between breakdowns $n-1$ and n. Then X_n consists of repair time plus remaining interarrival time. Hence all X_n have the same distribution and are independent.

   b. Elementary renewal theorem and/or strong law of large numbers for renewal processes: long run (expected) rate is $1/EX$, where $EX = T/2 + 1/\lambda$. So long run rate is $2\lambda/(\lambda T + 2)$.

   c. Regenerative process or alternating renewal process:
   E repair time/E cycle time $= \frac{T/2}{T/2+1/\lambda} = \lambda T/(\lambda T + 2)$.

5. a. No. The relation $ES_{N(t)+1} = \mu[m(t)+1]$ holds since $N(t)+1$ is a stopping time for the sequence $\{X_i\}$. But $N(t)$ is not (e.g. the event $N(t)=2$ also depends on the value of X_3), so we cannot use Wald to conclude that $E\sum_{i=1}^{N(t)} X_i = EXEN(t)$. Alternative: suppose that $ES_{N(t)} = \mu m(t)$ is true. Then the expectation of the renewal interval containing t would be $EX_{N(t)+1} = ES_{N(t)+1} - ES_{N(t)} = E\mu[m(t)+1] - \mu m(t) = \mu$. But this is in contradiction with the inspection paradox, which says that $EX_{N(t)+1} > \mu$ (unless X_i is some constant w.p. 1).

b. Using partial integration we find

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\begin{aligned}
Ee^{-sY} &= \int_0^\infty e^{-sx}\frac{d}{dx}P(Y \le x)dx \\
&= \mu^{-1}\int_0^\infty e^{-sx}P(X > x)dx \\
&= \mu^{-1}\int_0^\infty e^{-sx}(1-F(x))dx \\
&= -(\mu s)^{-1}e^{-sx}(1-F(x))|_0^\infty - (\mu s)^{-1}\int_0^\infty e^{-sx}dF(x) \\
&= (\mu s)^{-1} - (\mu s)^{-1}\phi(s) = (1-\phi(s))/s\mu.
\end{aligned}
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