LNMB EXAM Introduction to Stochastic Processes (ISP), September 30, 2014.

The exam consists of six exercises, each with several parts. For each part it is indicated between square brackets how many points can be earned for the item; the total number of points is 40.

Exercise 1 The discrete-time Markov chain with state space $S = \{1, 2, 3, 4, 5\}$, has transition matrix

$$P = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{5} & 0 & 0 & \frac{4}{5} \\ 0 & 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & \frac{2}{5} & 0 & 0 & \frac{3}{5} \end{pmatrix}.$$

At time 0 the Markov chain starts with probability 1 in state 1. Each visit to state 1 will cost 40 euro per time unit, to state 2 it will cost 35 euro, to state 3 it will cost 60 euro, to state 4 it will cost 25 euro, to state 5 it will cost 70 euro.

(a) [3pt] Determine the limiting distribution of the Markov chain.

(b) [2pt] Determine the total expected time the process will be in the transient states.

(c) [2pt] Determine the long-run expected costs per time unit.

Exercise 2 An insurance company receives false and real claims according to two independent Poisson processes with rates λ_f and λ_r, respectively.

(a) [1pt] What is the probability that in $[0, 1]$ exactly two claims will arrive?

(b) [1pt] What is the probability that, given that two claims arrive in $[0, 1]$, the first one is a real claim and the second one is a false claim?

(c) [1pt] What is the probability that in $[0, 1]$ the same number of false and real claims will arrive?

(d) [2pt] What are, given that two claims arrive in $[0, 1]$, the expected arrival instants of the first claim and the second claim?

Exercise 3 Let $\{X_n : n \geq 0\}$ be a branching process with immigration given by the equation

$$X_{n+1} = \sum_{i=1}^{X_n} Z_i^{(n)} + Y_{n+1}.$$

Here the random variables $Z_i^{(n)}$ are independent, identically distributed Bernoulli random variables, i.e., $\mathbb{P}(Z_i^{(n)} = 1) = p = 1 - \mathbb{P}(Z_i^{(n)} = 0)$, with $0 < p < 1$, and they are independent of X_n. The random variable $Z_i^{(n)}$ represents the number of children of the i-th individual in generation n. The random variable Y_{n+1} is a Poisson random variable with parameter λ, independent of all other random variables, and represents the number of immigrants in the $(n + 1)$-th generation. The process starts with 0 individuals, i.e., $X_0 = 0$.

(a) [2pt] What is the state space of the Markov chain $\{X_n : n \geq 0\}$? Is the Markov chain reducible or irreducible? Are the states of the Markov chain periodic or aperiodic? Motivate all your answers.

(b) [2pt] Show that $\mathbb{E}(X_{n+1}) = \lambda \sum_{k=0}^{n} p^k$, for $n = 0, 1, 2, \ldots$, and determine $\lim_{n \to \infty} \mathbb{E}(X_n)$.

(c) [2pt] Let $P_n(z) := \mathbb{E}(z^{X_n})$ be the probability generating functions of the random variables X_n, for $n = 0, 1, 2, \ldots$. Show that these probability generating functions satisfy the relation

$$P_{n+1}(z) = P_n(1 - p + pz) \cdot e^{\lambda(z-1)}.$$

(d) [2pt] Show that $P_{n+1}(z) = e^{\lambda(\sum_{k=0}^{n} p^k)(z-1)}$, for $n = 0, 1, 2, \ldots$, and determine $\lim_{n\to\infty} P_n(z)$.

(e) [1pt] What can we conclude from (d) for the transient distributions and the limiting distribution of the Markov chain $\{X_n : n \geq 0\}$?

Exercise 4 Jobs arrive at a computer processor according to a Poisson process with rate 1. The processor serves jobs one by one, according to the First Come First Served (FCFS) service discipline. Service times are exponentially distributed with rate 2. It has a buffer of size 2, i.e. if there are already two jobs in the buffer then any new arrival is lost. Hence, a new job can enter the buffer only if it finds the buffer upon arrival not full. All interarrival and service times are independent.

(a) [1 pt.] Explain why the process $\{X(t),\ t \geq 0\}$, with $X(t)$ the number of jobs at time t, is a CTMC and give the generator of the process or the transition rate diagram.

(b) [2 pt.] Derive the limiting distribution $p_n = \lim_{t\to\infty} \mathbb{P}[X(t) = n]$, $n = 0, 1, 2$, using the standard techniques from theory on CTMCs.

(c) [2 pt.] Calculate the mean amount of time it will take the system starting from state 2 to reach state 0.

(d) [2 pt.] Write the forward and backward Kolmogorov equations for $p_{00}(t)$.
(Do not try to solve them!)

Exercise 5 *Darkbulb* produces new light bulbs for which the company claims that they live "for ever". In order to support their claim, *Darkbulb* offers a warranty contract, in which for a nominal cost of 8 € per light bulb, *Darkbulb* promises to replace, free of cost, any light bulb that fails in the first 4 years of its life. The replaced light bulb comes with no warranty. On the other hand, if a light bulb fails after 4 years, a new one has to be purchased for the full price, according to the terms of the warranty. A new light bulb costs 70 €, and ISP students have established after painstaking measurements of numerous light bulbs that the bulbs themselves have an exponentially distributed lifetime with mean 10 years.

(a) [1 pt.] Assume that you never buy the warranty. Define a renewal process, say $N_0(t)$.

(b) [2 pt.] Show that $\mathbb{E}[S_{N_0(t)}] \neq \mathbb{E}[N_0(t)] * 10$.

(c) [2 pt.] Assume that you always buy the warranty. Define a renewal process, say $N_1(t)$.

(d) [2 pt.] Calculate the mean of the inter-renewal times for the $N_1(t)$ renewal process.

(e) [2 pt.] Is it worth buying the warranty?
(You may use that $e^{-4/10} \approx 0.67$.)

Exercise 6 In 1984 the country of Morocco in an attempt to determine the average amount of time that tourists spend in that country on a visit tried two different sampling procedures. In one, they questioned randomly chosen tourists as they were leaving the country; in the other, they questioned randomly chosen guests at hotels. (Each tourist stayed at a hotel.) The average visiting time of the 3000 tourists chosen from hotels was 17.8, whereas the average visiting time of the 12,321 tourists questioned at departure was 9.0.

(a) [3 pt.] Can you explain this discrepancy? Does it necessarily imply a mistake or can you describe a situation in which the measured data is reasonable?