FINAL EXAM TURBULENCE IN FLUIDS
20 April 2016, 8.30 - 10.30 (2 hours)
Three problems (each problem gives 15 points)

Remark 1: Answers may be written in English or Dutch.

Remark 2: Please write every problem on a separate sheet of paper.

Remark 3: In turbulent flows, the dissipation rate ε has units of velocity^2/time and the kinematic viscosity ν has units of $m^2 s^{-1}$.

Problem 1: Kelvin-Helmholtz instability (15 points)

Consider a two-layer stratified fluid, as in the left part of figure 1. The total thickness is H, the interface between the layers is at an arbitrary $z = h$, where $0 < h < H$. The upper and lower layer have densities ρ_1 and ρ_2, and velocities U_1 and U_2, respectively. We assume that $\rho_1 \approx \rho_2 \approx \rho_0$ (Boussinesq approximation).

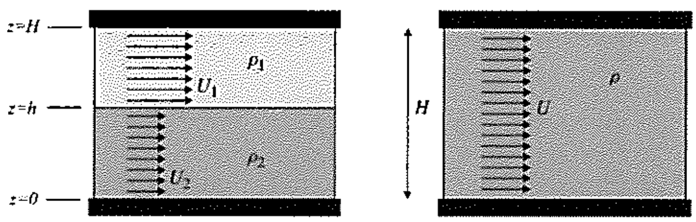

Figure 1: Stratified two-layer fluid (left) before mixing and uniform layer of fluid (right) after mixing.

(a) After complete mixing the density of the layer is ρ and the velocity is U. Give the expressions for ρ and U in terms of the defined variables. (2 points)

(b) Show that the loss of kinetic energy for complete mixing is equal to

$$\Delta \text{KE} = \frac{\rho_0 h}{2H}(H - h)(U_1 - U_2)^2$$

(5 points)

(c) By considering also the gain of potential energy, show that complete mixing is only possible if

$$\frac{(\rho_2 - \rho_1)gH}{\rho_0(U_1 - U_2)^2} \leq 1 \qquad (1)$$

Name the two effects that can stabilise or destabilise such a flow. (5 points)

(d) The instability sets in at the interface between the two layers in the form of interfacial waves. The stability criterion (1) can be expressed in terms of the wave number k of an interfacial wave. This wave is unstable if in a Boussinesq fluid

$$2(\rho_2 - \rho_1)g < \rho_0 k(U_1 - U_2)^2$$

Can you think of a completely stable situation in such a two-layer shear flow (explain you answer)? From the instability criterion, derive an expression for the wavelength of the longest unstable wave. What does this mean for the stability of a two-layer shear flow with equal densities $\rho_1 = \rho_2$? (3 points)

Problem 2: Self similar turbulent flows (15 points)

A class of free shear flows can be assumed to be statistically stationary and statistically two-dimensional. In this case, the mean flow is described by the so-called boundary layer equations consisting of the mean continuity equation and one mean momentum equation and depending on only two spatial coordinates. For the round jet, which is statistically stationary and ax-isymmetric, the boundary layer equations are most conveniently written in polar-cylindrical coordinates, i.e.

$$\frac{\partial\langle U\rangle}{\partial x} + \frac{1}{r}\frac{\partial\left(r\langle V\rangle\right)}{\partial r} = 0 \tag{2}$$

$$\langle U\rangle\frac{\partial\langle U\rangle}{\partial x} + \langle V\rangle\frac{\partial\langle U\rangle}{\partial r} = \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\langle U\rangle}{\partial r}\right) - \frac{1}{r}\frac{\partial}{\partial r}\left(r\langle uv\rangle\right). \tag{3}$$

This flow can be characterised by a velocity scale $U_0(x)$ and length scale $r_{1/2}(x)$, defined by $U_0(x) \equiv \langle U(x,0,0)\rangle$ and $\langle U(x, r_{1/2}, 0)\rangle = \frac{1}{2}U_0(x)$, respectively.

(a) Very often these type of flows can be considered *self-similar*. By referring to the quantities $\langle U\rangle/U_0(x)$ and $r/r_{1/2}(x)$, explain what this self-similarity assumption means for the round jet and what the meaning of $r_{1/2}(x)$ and $U_0(x)$ is. Furthermore, explain what happens if this self-similarity ansatz is applied to the boundary layer equations (2, 3) with respect to the solvability of these equations? (4 points)

We now assume that the self-similarity profile for the streamwise velocity is given by a function $\bar{f}(\xi)$ (where $\xi = r/r_{1/2}$), such that $\langle U(x,r,0)\rangle = U_0(x)\bar{f}(\xi)$. Similarly, the Reynolds stresses are self similar with function $\bar{g}(\xi)$, such that $\langle uv\rangle = U_0(x)^2\bar{g}(\xi)$. In the following problems, show that these self-similarity profiles together with the boundary layer equations imply that the jet spreads linearly.

(b) Show that the derivatives of $\langle U\rangle$ in a self-similar round jet are

$$\frac{r_{1/2}}{U_0}\frac{\partial\langle U\rangle}{\partial x} = \bar{f}\left(\frac{r_{1/2}}{U_0}\frac{dU_0}{dx}\right) - \xi\bar{f}'\left(\frac{dr_{1/2}}{dx}\right) \tag{4}$$

$$\frac{r_{1/2}}{U_0}\frac{\partial\langle U\rangle}{\partial r} = \bar{f}', \tag{5}$$

a prime denoting differentiation with respect to ξ. (4 points)

(c) Using the continuity equation (2), the mean lateral velocity is

$$\frac{\langle V\rangle}{U_0} = \xi\bar{f}\left(\frac{dr_{1/2}}{dx}\right) - \left(\frac{r_{1/2}}{U_0}\frac{dU_0}{dx} + 2\frac{dr_{1/2}}{dx}\right)\frac{1}{\xi}\int_0^{\xi}\hat{\xi}\bar{f}d\hat{\xi}, \tag{6}$$

where $\hat{r}$ and $\hat{\xi}$ are integration variables. By substituting the relations (4, 5, 6) into the boundary-layer momentum equation (3), and neglecting the viscous term, show that the boundary layer momentum equation can be written as

$$[\xi\bar{f}^2]\left\{\frac{r_{1/2}}{U_0}\frac{dU_0}{dx}\right\} - \left[\bar{f}'\int_0^{\xi}\hat{\xi}\bar{f}d\hat{\xi}\right]\left\{\frac{r_{1/2}}{U_0}\frac{dU_0}{dx} + 2\frac{dr_{1/2}}{dx}\right\} = [(\xi\bar{g})'], \tag{7}$$

where the terms in brackets [...] depend only on ξ and those in braces {...} depend only on x. (4 points)

(d) Explain why the two terms in braces {...} on the l.h.s. of equation (7) must be independent of x and therefore constant. Finally, by writing the two constant terms as

$$\frac{r_{1/2}}{U_0}\frac{dU_0}{dx} = C$$
$$\frac{r_{1/2}}{U_0}\frac{dU_0}{dx} + 2\frac{dr_{1/2}}{dx} = C + 2S$$

with constants C and S, show that the spreading rate of the jet is linear in x. (3 points)

Problem 3: The turbulent energy cascade (Total 15 points)

In 1922 Richardson invented the picture of the turbulent energy cascade, which he characterised by the following little poem:

Big whirls have little whirls
that feed on their velocity
and little whirls have lesser whirls
and so on to viscosity.

(a) Draw a sketch of the energy cascade in terms of spatial and temporal scales and characteristic velocities. Explain in particular the last alinea of Richardson's poem, and mention the quantities that determine the dissipation rate ε. (3 points)

(b) Formulate Kolmogorov's first similarity hypothesis, explain in which range of scales it applies, and derive from the hypothesis the Kolmogorov length, velocity and time scales η, u_η and τ_η, respectively. Are these large or small scales in the turbulent flow? Explain you answer. (4 points)

(c) Kolmogorov formulated his hypotheses in terms of the second order velocity structure function $D_{ij}(\boldsymbol{r}, \boldsymbol{x}, t)$. For homogenous and isotropic turbulence this second order tensor is fully determined by two scalar functions $D_{LL}(r,t)$ and $D_{NN}(r,t)$, and, as a consequence of the continuity equation, D_{NN} and D_{LL} are not independent of each other, i.e.:

$$D_{NN}(r,t) = D_{LL}(r,t) + \frac{1}{2}r\frac{\partial}{\partial r}D_{LL}(r,t). \tag{8}$$

As a consequence of the first similarity hypothesis, $D_{LL}(r,t) = (\varepsilon r)^{2/3}\hat{D}_{LL}(r/\eta)$, where $\hat{D}_{LL}(r/\eta)$ is a universal, non-dimensional function. Formulate Kolmogorov's second similarity hypothesis, explain in which range of scales it applies, and show the consequences for $D_{LL}(r,t)$ in that range. Use equation (8) to calculate $D_{NN}(r,t)$ in the range of scales where the second similarity hypothesis holds. (4 points)

(d) The *von Kàrmàn-Howarth equation*, expressed in terms of the velocity structure function becomes

$$\frac{3}{r^5}\int_0^r s^4 \frac{\partial D_{LL}(s,t)}{\partial t} ds = 6\nu \frac{\partial D_{LL}}{\partial r} - D_{LLL} - \frac{4}{5}\varepsilon r$$

Where is this equation based on? Explain how Kolmogorov argued that this equation, in the inertial subrange, quantifies the third order longitudinal structure function for stationary, isotropic and homogeneous turbulence as:

$$D_{LLL}(r,t) = \frac{4}{5}\varepsilon r.$$

(4 points)

END