MID-TERM EXAM TURBULENCE IN FLUIDS

6 March 2017, 10.00 - 12.00 (2 hours)

Three problems (giving a total of 45 points)

Remark 1: Answers may be written in English or Dutch.

Remark 2: Please write every problem on a separate sheet of paper.

Problem 1 (Total = 16 + 2 bonus points)

Consider the dynamical system given by

$$
\begin{aligned}
\frac{dA}{dt} &= AB - A, \\
\frac{dB}{dt} &= -A^2 - B + \gamma,
\end{aligned}
$$

with state vector $(A, B) \in \mathrm{R}^2$ and control parameter γ.

4 (a) (4P) Determine the steady states (or fixed points) $(\bar{A}, \bar{B})$ of the dynamical system for (i) $\gamma = 1/2$ and (ii) for $\gamma = 3/2$.

4 (b) (4P) Consider a representation of the bifurcation behavior of the dynamical system by drawing $\bar{A}$ versus γ. What type of bifurcation occurs at $\gamma = 1$?

4 (c) (4P) Show that the steady state $(\bar{A}, \bar{B})$ with $\bar{A} > 0$ is stable at $\gamma = 3/2$.

The steady state in c. is next slightly perturbed given an initial condition

$$
\begin{aligned}
A(t = 0) = A_0 &= \bar{A} + \tilde{A} \\
B(t = 0) = B_0 &= \bar{B} + \tilde{B}
\end{aligned}
$$

where the tildes indicate the perturbation.

4 (d) (4P) Sketch a typical trajectory, which develops from the initial state (A_0, B_0) in the (A, B) plane.

Consider now all steady states $\bar{A} > 0$ for $\gamma > 1$.

1/2 (e) (2 bonus points) For which value of γ will the behavior of trajectories as sketched under d. qualitatively change?

For problem 2: P.T.O.

Problem 2 (Total 17 points)

The Lorenz low-order model of thermal convection between two perfectly conducting and stress-free parallel plates, which are held at different temperature, consists of the following three equations:

$$\frac{dX}{dt} = \frac{a\mathrm{Pr}}{\pi(a^2+1)}Y - \pi^2(a^2+1)\mathrm{Pr}X \tag{1}$$

$$\frac{dY}{dt} = -2\pi^2 aXZ + \pi a\mathrm{Ra}X - \pi^2(a^2+1)Y \tag{2}$$

$$\frac{dZ}{dt} = 4\pi^2 aXY - 4\pi^2 Z, \tag{3}$$

where

$$\mathrm{Pr} \equiv \frac{\nu}{\kappa},\ \mathrm{Ra} \equiv \frac{g\alpha H^4\Gamma}{\nu\kappa},\ \text{and } a \equiv \frac{2H}{L}$$

are non-dimensional numbers. Symbols are defined as usual.

(a) (3P) The Lorenz model is physically most realistic at a Rayleigh number (Ra), which is slightly higher than the critical value for the instability of the state of rest to small perturbations. Explain why.

(b) (3P) The Lorenz model is also most realistic for large values of the Prandtl number (Pr). Explain why.

(c) (3P) Show that, in the limit of very large Pr, the system of three equations (1-3) with three unknowns can be approximated by a system of two equations with two unknowns of the form:

$$\frac{dY}{dt} = \frac{-2a^2}{\pi(a^2+1)^2}YZ + \frac{a^2 Ra}{\pi^2(a^2+1)^2}Y - \pi^2(a^2+1)Y \tag{4}$$

$$\frac{dZ}{dt} = \frac{4a^2}{\pi(a^2+1)^2}Y^2 - 4\pi^2 Z, \tag{5}$$

(d) (4P) One steady equilibrium solution, or fixed point, of eqs. 4 and 5 is $(Y, Z) = (0, 0)$. This fixed point corresponds to the state of rest in which only diffusive vertical transport of heat is possible. Perform a linear stability analysis of this fixed point and derive an expression for the critical Rayleigh number for the onset of convection.

(e) (4P) Determine the minimum critical Rayleigh number for the onset of convection. $R_{c.}\ \frac{27}{4}\pi^7$

Problem 3 (Total 12 points)

(a) (3P) What are the three principal ingredients of a chaotic solution to a system of first order non-linear differential equations, like the Lorenz (1963) equations?

For remainder of problem 3: next page

In the chaotic regime, a non-linear system, like the Lorenz model, has solutions whereby nearby trajectories in phase space separate at an approximate exponential rate according to

$$|\delta(t)| \simeq \delta_0 |e^{(\lambda t)}|$$

(see left panel of figure 1). The parameter λ is called the Lyapunov-exponent. The right panel of figure 1 shows a plot of $\ln(\delta)$ as a function of non-dimensional time corresponding to two solutions of the Lorenz model starting from slightly different initial conditions. In the first integration the initial condition is given by

$$X = X_0;\ Y = Y_0 + 0.1;\ Z = Z_0,$$

while in the second integration the initial condition is given by

$$X = X_0;\ Y = Y_0 + 0.2;\ Z = Z_0,$$

where (X_0, Y_0, Z_0) corresponds to the steady state of finite amplitude convection. This steady state (fixed point) is linearly unstable.

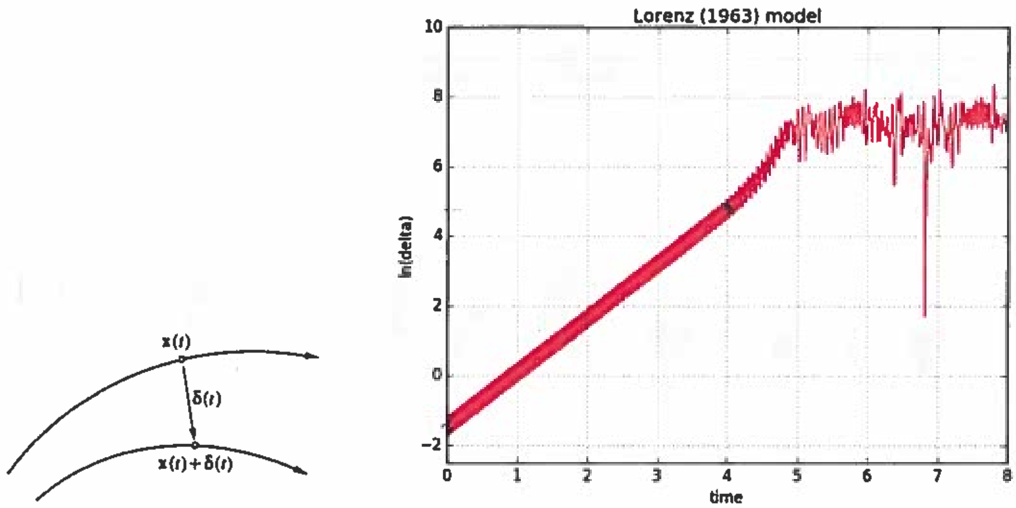

Figure 1: **Left:** The solutions of a non-linear system represented by two trajectories in phase space. The distance separating these solutions in phase space is given by δ. **Right:** The natural logarithm of δ (non-dimensional) as a function of non-dimensional time for two solutions of the Lorenz with slightly different initial conditions ($Ra = 28Ra_c$, $Pr = 10$ and $a = 2^{-1/2}$).

(b) (3P) Estimate the Lyapunov-exponent from the right panel of figure 1.

(c) (3P) Why does the curve in the right panel of figure 1 level off after t=5?

(d) (3P) If $\lambda > 0$, and depending on a predefined tolerance, the predictability in a non-linear system is limited to a certain time horizon. Suppose that a model predicts temperature in units of Kelvin. The initial error is 0.1 K. If we tolerate an error of 1 K in the predicted temperature we can on average predict 2 days ahead with this model. How much further can we predict the temperature with the same model if the initial error is 0.01 K?

END