FINAL EXAM TURBULENCE IN FLUIDS 2021/22

11 April 2022, 8.45 - 10.45 (2 hours)

Two problems (giving a total of 45 points)

Remark 1: Before you start, please read the problems carefully. For each problem you are expected to work for about 1 hour.

Remark 2: Please write your name and student number on every piece of paper.

Remark 3: Answers may be written in English or Dutch.

Remark 4: In Problem 2 you have to use your laptop. Please send your solutions of problem 2 as one single PDF-file by email to A.J.vanDelden@uu.nl before 11:00h on 11 April 2022. Don't forget to include your name in the document.

Problem 1: Self similar turbulent flows (total 21 points)

A class of free shear flows can be assumed to be statistically stationary and statistically two-dimensional. In this case, the mean flow is described by the so-called boundary layer equations consisting of the mean continuity equation and one mean momentum equation and depending on only two spatial coordinates.

In this problem we consider the ideal plane wake behind a cylinder (see Figure 1 below), which is similar to the free shear flows treated in class. There is a uniform stream of strength U_c in the x-direction, the cross-stream coordinate is y, and statistics are independent of the spanwise coordinate z (the cylinder is aligned with z). There is statistical symmetry about the plane $y = 0$. The continuity and boundary layer equation are given by

$$\frac{\partial \langle U\rangle}{\partial x} + \frac{\partial \langle V\rangle}{\partial y} = 0 \tag{1}$$

$$\langle U\rangle\frac{\partial \langle U\rangle}{\partial x} + \langle V\rangle\frac{\partial \langle U\rangle}{\partial y} = -\frac{\partial}{\partial y}\langle uv\rangle + \nu\frac{\partial^2 \langle U\rangle}{\partial y^2}. \tag{2}$$

This flow can be characterised by two velocity scales, the mean flow U_c and the characteristic velocity difference $U_s(x) = U_c - \langle U(x, 0, 0)\rangle$. A characteristic length scale $\delta(x)$ is defined by $\langle U(x, \pm\delta, 0)\rangle = U_c - \frac{1}{2}U_s(x)$.

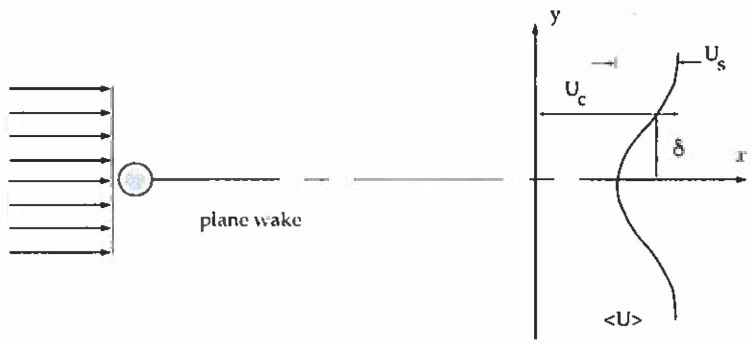

Figure 1: Sketch of the plane wake behind a cylinder aligned with the z-direction, with the characteristic velocity scales U_c and $U_s(x)$ and length scale $\delta(x)$.

(a) Explain the meaning of $U_s(x)$ and $\delta(x)$. The wake spreads and decays as x increases. What does that qualitatively mean for $\delta(x)$ and $U_s(x)/U_c$? (3P)

We now assume that the self-similarity profile for the streamwise velocity is given by a function $f(\xi)$ (where $\xi = y/\delta(x)$), which is called the velocity defect, $f(\xi) = [U_c - \langle U(x, y, 0)\rangle]/U_s(x)$. Similarly, the Reynolds stresses are self similar with function $g(\xi)$, such that $\langle uv\rangle = U_s(x)^2 g(\xi)$. In the following problems, the aim is to show that these self-similarity profiles together with the boundary layer equations imply that the wake spreads as $x^{1/2}$ and the velocity difference decays as $x^{-1/2}$.

(b) Show that for the plane wake, the boundary layer equation can be written as

$$\frac{\partial}{\partial x}[\langle U\rangle(U_c - \langle U\rangle)] + \frac{\partial}{\partial y}[\langle V\rangle(U_c - \langle U\rangle)] = -\frac{\partial}{\partial y}\langle uv\rangle. \tag{3}$$

Explain what has been neglected and why. (3P)

(c) Show that the boundary layer equation (3) implies that the momentum deficit flow rate $\dot{M}$ defined by

$$\dot{M} = \int_{-\infty}^{+\infty} [\rho\langle U\rangle(U_c - \langle U\rangle)]dy \tag{4}$$

is independent of x. (3P)

(d) Write the deficit momentum flow rate $\dot{M}$ in terms of the self similarity variables $f(\xi)$ and $g(\xi)$ and argue why this implies that the product $U_s(x)\delta(x)$ is also independent of x. (3P)

(e) In the far wake, the boundary layer equation can be approximated by

$$U_c\frac{\partial\langle U\rangle}{\partial x} = -\frac{\langle uv\rangle}{\partial y}, \tag{5}$$

which leads in terms of the self similarity variables $f(\xi)$ and $g(\xi)$ to

$$S(\xi f)' = -g', \tag{6}$$

where the spreading rate S is defined by $S = \frac{U_c}{U_s}\frac{d\delta}{dx}$. Explain why Eq. (6) implies that S is constant (independent of x). (3P)

(f) Finally use the results of item (d) and (e) to show that $U_s(x) \sim x^{-1/2}$ and $\delta(x) \sim x^{1/2}$. Assume that all integration constants vanish in the self-similar region. (3P)

(g) The boundary layer equation (5) in the wake still exhibits a closure problem. Explain how this problem can be overcome by a turbulent viscosity hypothesis. In particular, in the self similar region, show that the turbulent viscosity hypothesis amounts to

$$g = \hat{\nu}_T f', \tag{7}$$

and that the turbulent viscosity in the self-similar regions should be independent of x. You can use the integrated version of the relation (6), i.e. $S\xi f = -g$. (3P)

Problem 2: Convection model (total 24 points)

Problem 2: see next page

Exam Turbulence in Fluids, part 2, 11 April 2022 (1 hour)
(open book; laptop computer required)

We are going to study numerical solutions of the twelve-component spectral model of poloidal convection. This is a truncated (low-order) model of poloidal thermal convection between two horizontal perfectly conducting and stress-free horizontal boundaries held at different temperatures. The model-equations that govern the time-evolution of the Fourier coefficients of the vertical velocity and temperature corresponding to the Fourier modes (waves), which are taken inside the truncation, are (see section 36 of the lecture notes),

$$
\begin{aligned}
\frac{d\Theta_{111}}{dt} &= -\frac{3}{2}\pi W_{111}\Theta_{022} - \frac{3}{2}\pi W_{021}\Theta_{112} + \pi\Theta_{001}W_{112} - 2\pi\Theta_{002}W_{111} - 3\pi\Theta_{003}W_{112} + \mathrm{Ra}W_{111} - \pi^2\left(4a^2+1\right)\Theta_{111} \\
\frac{d\Theta_{021}}{dt} &= -3\pi W_{111}\Theta_{112} + \pi\Theta_{001}W_{022} - 2\pi\Theta_{002}W_{021} - 3\pi\Theta_{003}W_{022} + \mathrm{Ra}W_{021} - \pi^2\left(4a^2+1\right)\Theta_{021} \\
\frac{d\Theta_{112}}{dt} &= \frac{3}{2}\pi W_{021}\Theta_{111} + \frac{3}{2}\pi W_{111}\Theta_{021} + \pi\Theta_{001}W_{111} - 3\pi\Theta_{003}W_{111} - 4\pi\Theta_{004}W_{112} + \mathrm{Ra}W_{112} - \pi^2\left(4a^2+4\right)\Theta_{112} \\
\frac{d\Theta_{022}}{dt} &= 3\pi W_{111}\Theta_{111} + \pi\Theta_{001}W_{021} - 3\pi\Theta_{003}W_{021} - 4\pi\Theta_{004}W_{022} + \mathrm{Ra}W_{022} - \pi^2\left(4a^2+4\right)\Theta_{022} \\
\\
\frac{d\Theta_{001}}{dt} &= -4\pi W_{111}\Theta_{112} - 4\pi W_{112}\Theta_{111} - 2\pi W_{021}\Theta_{022} - 2\pi W_{022}\Theta_{021} - \pi^2\Theta_{001} \\
\frac{d\Theta_{002}}{dt} &= 8\pi W_{111}\Theta_{111} + 4\pi W_{021}\Theta_{021} - 4\pi^2\Theta_{002} \\
\frac{d\Theta_{003}}{dt} &= 12\pi W_{111}\Theta_{112} + 12\pi W_{112}\Theta_{111} + 6\pi W_{021}\Theta_{122} + 6\pi W_{022}\Theta_{021} - 9\pi^2\Theta_{003} \\
\frac{d\Theta_{004}}{dt} &= 16\pi W_{112}\Theta_{112} + 8\pi W_{022}\Theta_{022} - 16\pi^2\Theta_{004} \\
\\
\frac{dW_{111}}{dt} &= -\frac{3}{2}\pi W_{022}W_{111} - \frac{3}{2}\pi W_{021}W_{112} + \frac{4a^2\,\mathrm{Pr}}{4a^2+1}\Theta_{111} - \pi^2\,\mathrm{Pr}\left(4a^2+1\right)W_{111} \\
\frac{dW_{021}}{dt} &= -3\pi W_{112}W_{111} + \frac{4a^2\,\mathrm{Pr}}{4a^2+1}\Theta_{021} - \pi^2\,\mathrm{Pr}\left(4a^2+1\right)W_{021} \\
\frac{dW_{112}}{dt} &= 3\pi\frac{4a^2+1}{4a^2+4}W_{111}W_{021} + \frac{4a^2\,\mathrm{Pr}}{4a^2+4}\Theta_{112} - \pi^2\,\mathrm{Pr}\left(4a^2+4\right)W_{112} \\
\frac{dW_{022}}{dt} &= 3\pi\frac{4a^2+1}{4a^2+4}W_{111}^2 + \frac{4a^2\,\mathrm{Pr}}{4a^2+4}\Theta_{022} - \pi^2\,\mathrm{Pr}\left(4a^2+4\right)W_{022}
\end{aligned}
\qquad \textbf{(163)}
$$

The twelve-component model has "multiple equilibria", i.e. several steady state solutions for the same values of the externally imposed parameters, Ra, Pr and a. These solutions represent hexagonal convection cells with either upward motion (up-hexagons) or downward motion (down-hexagons) in the centre of the cell, or steady two-dimensional convection cells aligned along the x-axis (rolls).

The model equations are integrated in time with the fourth order accurate Runge-Kutta scheme to approximate the time-derivative. A set of 1000 integrations, lasting 25 time units, is performed for randomly differing initial conditions. The initial condition consists of adding a random number between -10 and +10 (negative or positive!) to the initial values (=zero) of the amplitudes, W_{111} and W_{021}, using the *Numpy* random number generator. All the other ten amplitudes are zero initially.

For Pr=10, Ra=5000 and $a=1/(2\sqrt{2})$ the model finds a steady state solution within the prescribed length of time of the run (25 time units). The set of 1000 solutions consists of a collection of the following three basic steady solutions: rolls (W_{021}=6.99), down hexagons (W_{021}=3.86) and up-hexagons (W_{021}=-3.86), where rolls are slightly more probable than up-hexagons or down-hexagons.

The scatter plot in **figure 1** shows the final state of the 1000 model integrations. The right panel is a close-up of the left panel. The black circle with a coloured cross represents a steady state, in terms of W_{111} and W_{021}, where a red cross corresponds to a down hexagon, a blue cross corresponds to an up-hexagon and a green cross corresponds to rolls aligned along the x-axis. The steady state of rest ($W_{111}=W_{021}=0$) is not indicated explicitly because it is linearly unstable at Ra=5000 and $a=1/(2\sqrt{2})$. The **red dots** represent the initial values of W_{111} and W_{021} of runs having a **down-hexagon** as final steady state (at t=25). The **blue dots** represent the initial values of W_{111} and W_{021} of runs having an **up-hexagon** as final stable steady state. The **green dots** represent the initial values of W_{111} and W_{021} of runs having a **roll convection pattern** as final stable steady state.

The 1000 integrations lead to rolls in 388 cases, an up-hexagon in 305 cases and a down-hexagon in 307 cases. The fixed points associated with steady state hexagons lie very close to the edge of their respective basins of attraction, bordering the basins of attraction of rolls.

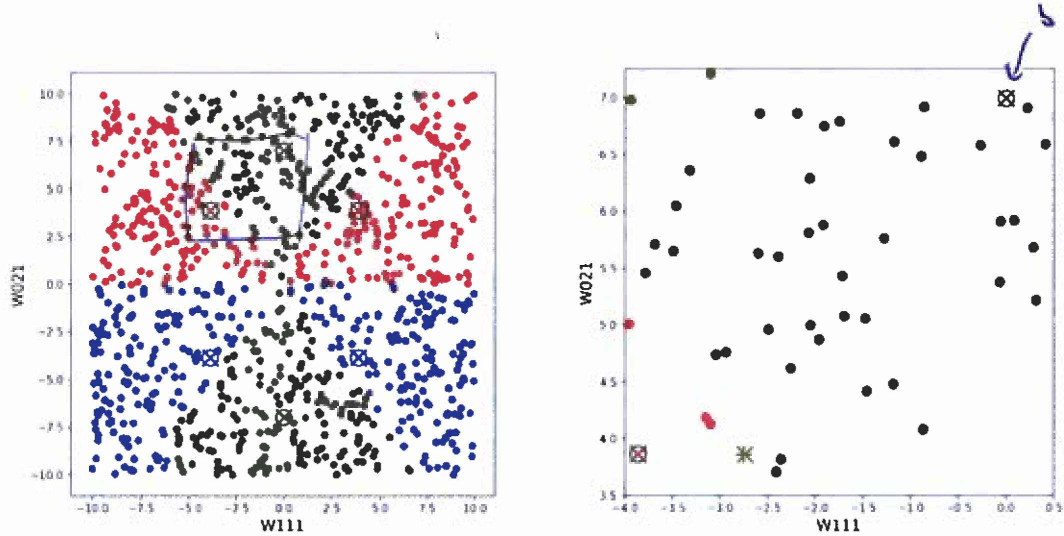

Figure 1. Left panel: basins of attraction of **rolls (green)**, **down-hexagons (red)** and **up-hexagons (blue)** (see the text for further explanation) in the W_{111}-W_{021} plane with all other coefficients equal to zero initially. The right panel is a close-up of the left panel, showing in addition the *initial state* of an extra **pair of integrations** starting at, respectively, $(W_{111}, W_{021})=(-2.75, 3.86)$ (the **red plus**) and $(W_{111}, W_{021})=(-2.74, 3.86)$ (the **green cross**). All other amplitudes are zero initially. The *final state* in the first case is $(W_{111}, W_{021})=(-3.86, 3.86)$ (a down-hexagon) and in the second case the *final state* is $(W_{111}, W_{021})=(0, 6.99)$ (a roll).

To answer the eight questions below you will need the Python-script of the twelve-component model (eq. set 163) (the file, `PoloidalConvectionModel[vrs1].py`, on Blackboard (Lecture 10)).

(a) Perform the additional **pair of model-integrations**, or runs, which are referred to in the caption of **figure 1**. The runs should last at least 1.5 non-dimensional time units. Plot W_{111} and W_{021} as a function of time for both runs. Shortly discuss the result (qualitatively).

(b) Show that the roll solution, indicated by the **green cross & black circle** in the right panel of **figure 1**, with $(W_{111}, W_{021})=(0, 6.99)$, is also a solution of the Lorenz model (eqs. 87-89 in the lecture notes) for $a_x=1/\sqrt{2}$.

(c) *Plot* a measure of the separation "distance" in twelve-dimensional phase space between the two time-dependent solutions, determined in part **(a)**, as a function of time.

(d) Does this separation "distance" behave in accord with Lyapunov's hypotheses (see **section17** of the lecture notes)? Explain why. Determine the associated Lyapunov exponent, if possible.

(e) When Ra=5000, the fluid layer gains heat by conduction through the lower solid boundary and looses heat by conduction through the upper solid boundary. Heat is transferred vertically through the fluid both by molecular conduction and by convection. Demonstrate that steady state two-dimensional convection cells (rolls) are more efficient at transferring heat upwards than steady state down-hexagons at Pr=10, Ra=5000 and $a=1/(2\sqrt{2})$, according to the 12-component model (eqs. 163).

(f) Do you get a strange attractor, similar to Lorenz's "butterfly" (**figure 10** of the lecture notes), when you repeat the **pair of model-integrations** of part **(a)** for $Ra=28Ra_c=18410$ (instead of Ra=5000)?

(g) *Plot* a measure of the separation "distance" in twelve-dimensional phase space between the pair of time-dependent solutions in this case, as in part **(c)**.

(h) Is the solution *more sensitive to initial conditions* at Ra=18410 than at Ra=5000? If so, why?