# Final exam, Mathematical Modelling (WISB357)

Wednesday, 1 February 2017, 9.00-12.00, BBG 161

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- Write your name on each page you turn in, and additionally, on the first page, write your student number and the *total number of pages submitted.*
- For each question, motivation your answer.
- You may make use of results from previous subproblems, even if you have been unable to prove them.
- For this midterm exam you are allowed to bring an A4 with notes on one side. You may not consult solutions to the problems, nor use a graphical calculator or smart phone.

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`Solution.` In small type-font letters.

`Scoring.`

Maximum possible points per part is shown in the margin.

Your score is the total number of points received divided by 3.

**Problem 1**. This problem concerns the "reaction-diffusion" equation

$$u_t = Du_{xx} - cu, \quad \text{for} \quad \begin{cases} -\infty < x < \infty, \\ 0 < t, \end{cases}$$

with the initial condition

$$u(x, 0) = f(x).$$

Assume $c$ and $D$ are positive constants.

(a) Using the Fourier Transform, find the solution of the above problem.

5 Solution. Let $U_k(t) = \mathcal{F}u(x,t)$, $k \in \mathbb{Z}$ be the Fourier transform of $u(x,t)$ in space, and $F_k = \mathcal{F}f(x)$. Then transforming the partial differential equation and initial condition yields

$$\dot{U}_k = -(k^2 D + c)U_k, \quad U_k(0) = F_k, \qquad k \in \mathbb{Z}.$$

The exact solution is

$$U_k(t) = \exp\left[-(k^2 D + c)t\right] F_k = e^{-ct}\left[e^{-k^2 Dt} F_k\right]$$

Applying the inverse transform from Table 4.1 of the textbook to the terms in brackets we find the solution

$$u(x,t) = \frac{e^{-ct}}{2\sqrt{\pi Dt}} \int_{-\infty}^{\infty} \exp\left[\frac{(x-s)^2}{4Dt}\right] f(s)\, ds.$$

(b) Show that the problem can also be solved by applying the transformation $u = ve^{at}$, for a carefully chosen constant $a$, followed by using known solution of the diffusion equation.

5 Solution. Let $u(x,t) = v(x,t)e^{at}$, hence

$$u_t = v_t e^{at} + ave^{at}, \quad u_{xx} = v_{xx}e^{at}, \quad \text{and} \quad u(x,0) = v(x,0) = f(x).$$

Substituting these into the partial differential equation and initial condition yields

$$\begin{aligned} v_t e^{at} + ave^{at} &= Dv_{xx}e^{at} - cve^{at}, \qquad v(x,0) = f(x) \\ v_t e^{at} + (a+c)ve^{at} &= Dv_{xx}e^{at}. \end{aligned}$$

Choosing $a = -c$ eliminates the second term. Dividing by $e^{at}$ leaves the diffusion equation

$$v_t = Dv_{xx}, \qquad v(x,0) = f(x),$$

for which the solution is

$$v(x,t) = \frac{1}{2\sqrt{\pi D t}} \int_{-\infty}^{\infty} \exp\left[\frac{(x-s)^2}{4Dt}\right] f(s)\, ds.$$

To get $u(x,t)$ we just multiply this by $e^{at} = e^{-ct}$.

**Problem 2**. Suppose "traffic" is governed by the Burgers equation

$$\rho_t + \rho\rho_x = 0,$$

with initial condition

$$\rho(x,0) = \begin{cases} 0, & x \le -1, \\ \frac{1}{2}(1+x), & -1 < x < 1, \\ 1, & 1 \le x. \end{cases}$$

(a) Sketch the characteristics in the $(x,t)$-plane.

1 Solution. The equation for constant $\rho$ is

$$\rho_t + \frac{dX}{dt}\rho_x = 0.$$

Evidently, $\frac{dX}{dt} = \rho(X(0),0)$. That is, the characteristic emanating from $X(0) = x$ has slope $\rho(x,0)$. The characteristics emanating from $x < -1$ are vertical lines, perpendicular to the $x$-axis. Those emanating from $x > 1$ are lines with slope 1 to the right. The characteristics between -1 and 1 vary in angle from 90° to 45°, with angle decreasing linearly from left to right.

(b) Find the solution, $\rho(x,t)$, using the method of characteristics.

6 Solution. The characteristic passing through a point $(x_1,t_1)$ is a line satisfying $x_1 = x_0 + t_1\rho(x_0,t_0)$. Along this line, the density is constant: $\rho(x_1,t_1) = \rho(x_0,t_0)$.
For $x_0 < -1$, we find $x_1 = x_0$, hence $\rho(x_1,t_1) = 0$.
For $x_0 > 1$, we find $x_1 = x_0 + t_1$ and $\rho = 1$.
For $-1 \le x_0 \le 1$ we find $x_1 = x_0 + \frac{t_1}{2}(1+x_0)$. Solving this expression for $x_0$ yields

$$x_0 = \frac{x_1 - \frac{t_1}{2}}{1 + \frac{t_1}{2}}, \qquad \rho(x_1,t_1) = \rho(x_0,0) = \frac{1+x_1}{2+t_1}$$

with $\rho(x_1,t_1) = \frac{1}{2}(x_0+1)$. Summarizing, the solution is

$$\rho(x,t) = \begin{cases} 0, & x < -1, \\ \frac{1+x}{2+t}, & -1 \le x \le 1+t, \\ 1, & x > 1+t. \end{cases}$$

(c) Find the points in the $(x,t)$-plane where $\rho = 1/3$.

2 Solution. In particular along this characteristic, $\rho(x_0,0) = 1/3$. This relation can be solved for $x_0$ to obtain

$$\rho(x_0,0) = \frac{1}{3} = \frac{1}{2}(1+x_0) \quad \Rightarrow \quad x_0 = -\frac{1}{3}.$$

The equation for the characteristic is

$$x = x_0 + \rho(x_0,0)t = -\frac{1}{3} + \frac{1}{3}t.$$

Along this line $\rho$ is constant and equal to $1/3$.

(d) Show that $v = \frac{1}{2}\rho$. Determine the flux $J$.

1 Solution. The transport equation is

$$\rho_t + J(\rho)_x = 0 = \rho_t + J'(\rho)\rho_x.$$

Evidently, $J'(\rho) = \rho$, and thus

$$J(\rho) = \frac{\rho^2}{2}.$$

Since $J(\rho) = \rho v(\rho)$, it follows that $v(\rho) = \rho/2$.

**Problem 3**. A linearly elastic bar is made of two different materials, and before being stretched it occupies the interval $0 \le A \le \ell_0$. Also, before being stretched, for $0 \le A < A_0$, the modulus and density are $E = E_L$ and $R = R_L$, while for $A_0 < A < \ell_0$ they are $E = E_R$ and $R = R_R$. Both $R_L$ and $R_R$ are constants. (*Hint:* It is useful to define separate functions $U_L(A)$, $U_R(A)$, $T_L(A)$, $T_R(A)$, etc. on the left and right parts of the domain.)

(a) The requirements at the interface, where $A = A_0$, are that the displacement and stress are continuous. Express these requirements mathematically, using one-sided limits.

2 Solution. Denote the displacement and stress functions by $U_L(A)$ and $T_L(A)$ for $A < A_0$ and $U_R(A)$ and $T_R(A)$ for $A > A_0$. The boundary conditions are:

$$\lim_{A\to A_0^-} U_L(A) = \lim_{A\to A_0^+} U_R(A), \qquad \lim_{A\to A_0^-} T_L(A) = \lim_{A\to A_0^+} T_R(A).$$

(b) Suppose the bar is stretched and the boundary conditions are $U(0,t) = 0$ and $U(\ell_0,t) = \ell - \ell_0$. Assume there are no body forces. Find the steady state solution for the density, displacement and stress.

8 <u>Solution.</u> The steady state elastic problem is

$$0 = \partial_A T(A) = \partial_A (E \partial_A U(A)), \qquad U(0) = 0, \quad U(\ell_0) = \ell - \ell_0.$$

This equation holds on each interval $0 \leq A \leq A_0$ and $A_0 \leq A \leq \ell_0$, with the additional boundary conditions at $A_0$. Consequently, we obtain the following system of differential equations:

$$\begin{array}{lll} E_L \frac{\partial^2 U_L}{\partial A^2} = 0, & U_L(0) = 0, & U_L(A_0) = U_R(A_0), \\ E_R \frac{\partial^2 U_R}{\partial A^2} = 0, & U_R(\ell_0) = \ell - \ell_0, & E_L U_L'(A_0) = E_R U_R'(A_0). \end{array}$$

Both displacements are linear functions in $A$:

$$U_L(A) = \alpha_L A + \beta_L, \qquad U_R(A) = \alpha_R A + \beta_r.$$

The boundary conditions at $A = 0$ and $A = \ell_0$ imply

$$U_L(0) = 0 = \beta_L, \qquad U_R(\ell_0) = \ell - \ell_0 = \alpha_R \ell_0 + \beta_R,$$

from which it follows that $\beta_R = \ell - (1 + \alpha_R)\ell_0$. The conditions at the interface become

$$\begin{array}{lll} U_L(A_0) = U_R(A_0) & \iff & \alpha_L A_0 = \alpha_R A_0 + \ell - (1 + \alpha_R)\ell_0, \\ T_L(A_0) = T_R(A_0) & \iff & E_L \alpha_L = E_R \alpha_R. \end{array}$$

This yields a linear system of equations for $\alpha_L$ and $\alpha_R$ that can be solved by substitution to find

$$\alpha_L = \frac{E_R(\ell - \ell_0)}{A_0(E_R - E_L) + E_L \ell_0}, \qquad \alpha_R = \frac{E_L(\ell - \ell_0)}{A_0(E_R - E_L) + E_L \ell_0},$$

from which the solutions are defined:

$$\begin{array}{lll} U(A) & = & \begin{cases} \alpha_L A, & A \leq A_0, \\ \alpha_R (A - \ell_0) + \ell - \ell_0, & A > A_0, \end{cases} \\ T(A) & = & E_L \alpha_L = E_R \alpha_R, \\ R(A) & = & \begin{cases} \frac{R_L}{1+\alpha_L}, & A < A_0, \\ R(A) = \frac{R_R}{1+\alpha_R}, & A > A_0. \end{cases} \end{array}$$