<u>Final exam, Mathematical Modelling</u> (WISB357)

Tuesday, 30 Jan 2018, 13.30-16.30, BBG 0.23

- Write your name on each page you turn in, and additionally, on the first page, write your student number and the *total number of pages submitted.*
- For each question, motivation your answer.
- You may make use of results from previous subproblems, even if you have been unable to prove them.
- For this exam you are allowed to bring an A4 with notes on both sides. You may not consult solutions to the exercises, nor use a graphical calculator or smart phone.

<u>Solution.</u> In small type-font letters.

<u>Scoring.</u>

Maximum possible points per part is shown in the margin.

Your score is the total number of points received divided by 3.

<u>Problem 1</u>. The relative air speed $v(x)$ (with units m/s) at a height $x > 0$ above the wing of an airplane flying at constant speed V_0 (m/s) is modelled by the differential equation:

$$\rho V_0 \frac{\partial v}{\partial x} + \mu \frac{\partial^2 v}{\partial x^2} = 0,$$

where $\rho > 0$ is the constant density (kg/m^3) and $\mu > 0$ is the viscosity parameter with units $kg/(m \cdot s)$. In a coordinate system fixed to the wing, the boundary conditions are

$$v(0) = 0, \qquad v(L) = V_0,$$

where L is a given height (in m), far enough from the airplane to neglect its influence.

(a) Nondimensionalize the equation and boundary conditions, using L and V_0 to rescale x and v, respectively. Show that you obtain a dimensionless parameter $Re = \rho V_0 L/\mu$, the "Reynolds number".

[2] <u>Solution.</u> Let $v = V_0\widetilde{v}$ and $x = L\widetilde{x}$. Then $\frac{d}{dx} = \frac{1}{L}\frac{d}{d\widetilde{x}}$. The differential equation becomes

$$\frac{\rho V_0}{L}\widetilde{v}'(\widetilde{x}) + \frac{\mu V_0^2}{L^2}\widetilde{v}''(\widetilde{x}).$$

Dividing by the first coefficient, introducing the Reynolds number, and dropping the tildes gives

$$v'(x) + \frac{1}{Re}v''(x) = 0.$$

The boundary conditions become $v(0) = 0$, $v(1) = 1$.

(b) The Reynolds number is typically very large. Let $\varepsilon = Re^{-1} \ll 1$, and construct a two-term outer expansion for $v(x)$. Use it to satisfy the boundary condition at $x = L$.

[2] Solution. The differential equation becomes $v'(x) + \varepsilon v''(x) = 0$. Insert the asymptotic expansion $v(x) = v_0(x) + \varepsilon v_1(x) + \cdots$. The order $\mathcal{O}(1)$ term is

$$v_0'(x) = 0,$$

We can satisfy the boundary condition $v_0(1) = 1$ to yield $v_0(x) = 1$. However this does not satisfy the other boundary condition $v_0(0) = 0$. The order $\mathcal{O}(\varepsilon)$ term satisfies

$$v_1'(x) + v_0''(x) = 0,$$

from which follows that $v_1(x)$ is a constant function that needs to satisfy the boundary conditions $v_1(0) = v_1(1) = 0$. Consequently, we find $v_1(x) = 0$. In fact, all other terms are identically zero and the outer solution is just $v(x) = v_0(x) = 1$.

(c) Construct a one-term inner expansion.

[2] Solution. Introduce a rescaling $\bar{x} = \varepsilon^{-\gamma} x$ near $x = 0$. The derivative rescales as $d/dx = \varepsilon^{-\gamma} d/d\bar{x}$. The differential equation becomes

$$\varepsilon^{-\gamma} V' + \varepsilon^{1-2\gamma} V'' = 0$$

Choosing $\gamma = 1$ makes these terms of equal order in ε, leaving

$$V' + V'' = 0$$

Integrate once to obtain $V' + V = c$, c a constant. Using an integrating factor we find the solution

$$V(\bar{x}) = e^{-\bar{x}} V(0) + c(1 - e^{-\bar{x}}) = c(1 - e^{-\bar{x}}),$$

where the last equality follows from the initial condition $V(0) = 0$.

(d) Use the matching condition to construct a one-term composite solution.

[2] Solution. The matching condition requires $\lim_{\bar{x}\to\infty} V(\bar{x}) = \lim_{x\to 0} v_0(x) = 1$. Consequently, we need $c = 1$. The composite solution is

$$v(x) \approx v_0(x) + V(x/\varepsilon) - v_0(0) = 1 - e^{-x/\varepsilon}.$$

(e) An airplane manufacturer can use a simplified model outside of the "boundary layer" which is defined as the region where $v(x) < 0.99V_0$. How thick is the boundary layer as a function of ε?

[2] Solution. The boundary layer is the region $0 \le x < \delta$, where

$$v(\delta) = 0.99V_0.$$

In terms of the rescaled variables, $\widehat{\delta} = \delta/L$ satisfies

$$v(\widehat{\delta}) = 0.99 = 1 - e^{-\widehat{\delta}/\varepsilon}.$$

We find:

$$e^{-\widehat{\delta}/\varepsilon} = 0.01 \quad \Rightarrow \quad \widehat{\delta} = \varepsilon \ln 100$$

Consequently, the boundary layer thickness is $\delta = \varepsilon L \ln 100$.

Problem 2. Analysis of traffic in a section of highway within a distance π kilometers of a tunnel (i.e. $x \in [-\pi, \pi]$) has shown that density perturbations $\rho(x,t)$ to the otherwise steady flow evolve according to the relation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} J(\rho) = 0, \qquad J(\rho) = \frac{\rho^2}{2}.$$

During a given morning rush hour, the perturbation is observed to be

$$\rho(0,x) = \rho_0(x) = -\sin x.$$

Answer the following questions:

(a) What are the velocity function $v(\rho)$ and wave speed $c(\rho)$ that hold for this perturbation?

[2] Solution. The velocity and wave speed are related to the flux function by

$$J(\rho) = \rho v(\rho), \qquad c(\rho) = J'(\rho),$$

respectively. Apparently, $v(\rho) = \rho/2$ and $c(\rho) = \rho$.

(b) Sketch the characteristics and describe how the density perturbation evolves. (*Hint:* Here it is helpful to consider what happens to the characteristics emanating from points x_0 small enough that the approximation $\sin(x_0) \approx x_0$ holds.)

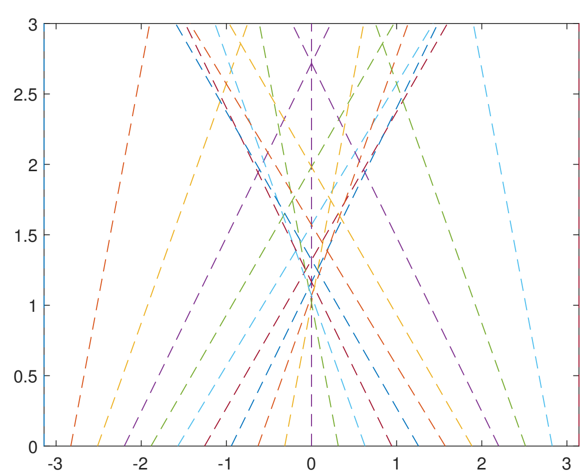
[4] Solution. The characteristic curves are

$$X(t) = x_0 + c(\rho_0(x_0))t = x_0 + \rho_0(x_0)t.$$

Due to the initial condition, all characteristics on $x_0 < 0$ have positive slope and those on $x > 0$ have negative slope. The characteristics will intersect and form a shock wave. For $\sin x_0 \approx x_0$, the characteristics approximately satisfy

$$X(t) = x_0 - \sin(x_0)t \approx x_0(1-t)$$

These all coalesce at $x = 0$ at time $t = 1$. For larger $|x_0|$ the characteristics intersect at a later time.



(c) What is the speed of the resulting shock wave?

[4] Solution. The Rankine-Hugoniot condition yields shock speed

$$s'(t) = \frac{1}{\rho_R - \rho_L} \int_{\rho_L}^{\rho_R} c(\rho)\, d\rho = \frac{1}{2} \frac{\rho_R^2 - \rho_L^2}{\rho_R - \rho_L} = \frac{\rho_R + \rho_L}{2}$$

Since the initial density ρ_0 is an odd function, we have $\rho_R = -\rho_L$. Hence, $s'(t) = 0$. The shock wave is stationary.

Problem 3. A manufacturer of bungee cords has developed a new cord for which the elasticity, expressed in terms of Young's modulus, varies with length according to $E(A) = (A/\ell_0)^{-2}$ for a cord of length ℓ_0, cross-sectional area σ and constant density R_0. To a good approximation, the bungee cord is linearly elastic $T(A) = E(A)\partial U/\partial A$ where $U(A,t)$ is the displacement function. The momentum equation for the motion of the bungee cord is expressed in material coordinates as:

$$R_0 \frac{\partial^2 U}{\partial t^2} = gR_0 + \frac{\partial T}{\partial A}.$$

Suppose a student of mass $M > 0$ is fastened to the end of the bungee cord at $A = \ell_0$. The other end at $A = 0$ is attached to a high bridge, and the student jumps off and bounces around awhile until he reaches a steady state $\partial U/\partial t \equiv \partial^2 U/\partial t^2 \equiv 0$ (due to air friction, apparently).

(a) State the boundary condition that holds for the stress $T(A)$ at $A = \ell_0$ and solve the differential equation for the stress along the cord $T(A)$, $0 \le A \le \ell_0$.

[4] Solution. The stress at $A = \ell_0$ is the weight of the student divided by the cross-sectional area: $T(\ell_0) = Mg/\sigma$. At steady state the stress satisfies:

$$\frac{dT}{dA} = -gR_0.$$

Integrating this expression from A to ℓ_0 gives

$$\int_A^{\ell_0} \frac{dT(a)}{da}\, da = \int_A^{\ell_0} -gR_0\, da \quad \Rightarrow \quad T(\ell_0) - T(A) = -gR_0(\ell_0 - A).$$

Noting the boundary condition,

$$T(A) = \frac{Mg}{\sigma} + gR_0(\ell_0 - A).$$

(b) State the boundary condition on the displacement $U(A)$ at $A = 0$ and solve for $U(A)$ and the equilibrium length $\ell = \ell_0 + U(\ell_0)$.

[4] Solution. At $A = 0$ there is no displacement, so $U(0) = 0$. Since $T(A) = E(A)U_A$, the differential equation for the displacement is

$$E(A)\frac{\partial U}{\partial A} = \frac{Mg}{\sigma} + gR_0(\ell_0 - A).$$

The displacement is

$$U(A) = U(0) + \int_0^A \left(\frac{Mg}{\sigma} + gR_0\ell_0\right)\frac{a^2}{\ell_0^2} - gR_0\frac{a^3}{\ell_0^2}\, da = \left(\frac{Mg}{\sigma} + gR_0\ell_0\right)\frac{A^3}{3\ell_0^2} - gR_0\frac{A^4}{4\ell_0^2}$$

The equilibrium length is

$$\ell = \left(1 + \frac{Mg}{3\sigma}\right)\ell_0 + \frac{gR_0}{12}\ell_0^2$$

(c) Note that $E(A)$ becomes unbounded as $A \to 0$. If the stress is to be finite at $A = 0$, what additional boundary condition should hold on the displacement at $A = 0$? Does your solution satisfy this condition? What does this condition imply about the stiffness or stretchability of the new bungee cord near $A = 0$?

[2] Solution. Finite stress as $A \to 0$ implies

$$\lim_{A \to 0} T(A) = \lim_{A \to 0} E(A) U_A < \infty.$$

For $E(A) = A^2/\ell_0^2$, this means

$$\lim_{A \to 0} \frac{U_A}{A^2} < \infty,$$

which (by l'Hôpital's rule) implies that the function $U(A)$ should satisfy the conditions

$$\lim_{A \to 0} U_A(0) = \lim_{A \to 0} U_{AA}(0) = 0.$$

This clearly holds for our solution. The bungee cord is very stiff near $A = 0$ since the rate of deformation is zero, even under loading.