UNIVERSITY OF AMSTERDAM

Faculty of Science

Exam

Asymptotic Statistics
MasterMath SFM (MSc.)

Final exam
Date: Wednesday 4 January 2017
Time: 10.00-13.00

Number of pages: 4 (including front page)
Number of questions: 3
Maximum number of points: 75
For each question is indicated how many points it is worth.

BEFORE YOU START

- Check if your version of the exam is complete.
- Write down **your name, student ID number**, and if applicable the **version number** on **each sheet** that you hand in. Also **number the pages**.
- Your **mobile phone** has to be switched off and be put in your coat or bag. Your **coat and bag** should be on the ground.
- **Tools allowed**: None.. Other tools are not allowed.

PRACTICAL MATTERS

- The first 30 minutes you are not allowed to leave the room, not even to visit the toilet.
- 15 minutes before the end, you will be warned that the time to hand in is approaching.
- If applicable, fill out the evaluation form at the end of the exam.
- You are obliged to identify yourself at the request of the examiner (or his representative) with a proof of your registration and a valid ID.
- During the examination it is not permitted to visit the toilet, unless the invigilator gives permission to do so.

Good luck!

Problem 1 *(Limit distributions)*

For the first half of this problem, let $X_n \sim \text{Bin}(n, p_1)$ and $Y_n \sim \text{Bin}(n, p_2)$ (with $0 < p_1, p_2 < 1$ and $n \geq 1$) and assume that X_n and Y_n are independent.

a. *(10 points)*
Find sequences (r_n) and (q_n) in $\mathbb{R}$ such that the sequence $r_n(X_n/Y_n - q_n)$ converges in distribution to a normal distribution. What are the mean and variance of the limiting normal distribution?

b. *(5 points)*
For given confidence level $0 < \alpha < 1$, determine an asymptotic confidence interval for the fraction $\theta = p_1/p_2$.

For the second half of this problem, let $X_1, \ldots, X_n$ form an *i.i.d.* sample from a distribution with density $f(x) = 2x$ with $x \in [0, 1]$. Denote $X_{(n)} = \max_{1 \leq i \leq n} X_i$.

c. *(10 points)*
Show that $Z_n = 2n(1 - X_{(n)})$ converges in distribution to a standard exponential distribution.

d. *(5 points)*
Determine sequences (a_n) and (b_n) in $\mathbb{R}$ such that $a_n(X_{(n)} \log X_{(n)} - b_n)$ converges in distribution to a (tight but non-degenerate) limit distribution. Which limit distribution?

Problem 2 *(A transformed exponential distribution)*

Let $X_1, X_2, \ldots$ be *i.i.d.* non-negative real-valued random variables with single-observation distribution P_{μ_0} and Lebesgue density $p_{\mu_0} : \mathbb{R} \to [0, \infty)$ for some $\mu_0 > 0$, with $p_\mu(x) = 0$ for $x < 0$, and

$$p_\mu(x) = 2\mu\, x\, e^{-\mu x^2},$$

for $x \geq 0$ and $\mu > 0$. A change of variables $Z = X^2$ leads to $Z \sim \text{Exp}(\mu)$.

Hint: you may use the following integrals,

$$\int_0^\infty x^2 e^{-x^2}\, dx = \frac{\sqrt{\pi}}{4}, \int x^3 e^{-x^2}\, dx = -\frac{e^{-x^2}(x^2+1)}{2} + C, \int x^5 e^{-x^2}\, dx = -\frac{e^{-x^2}(x^4+2x^2+2)}{2} + C,$$

a. *(5 points)*
Find the maximum-likelihood estimator $\hat{\mu}_n$ for μ_0 based on the first n sample points $X_1, \ldots, X_n$.

b. *(5 points)*
Calculate the expectation $E_\mu X_1^2$ of the second moment for a single observation and show that $\hat{\mu}_n$ is a consistent estimator sequence for μ_0.

c. *(5 points)*
Calculate the Fisher information I_μ for a single observation X_1, show that $\mu \mapsto I_\mu$ is continuous and that $I_\mu > 0$ for all $\mu > 0$.

d. *(5 points)*
Show that, for any $x > 0$, the map $\mu \mapsto \log p_\mu(x)$ is Lipschitz in a neighbourhood of μ_0. In other words, prove that for some $\epsilon > 0$ and any $\mu_1, \mu_2 > 0$ such that $|\mu_1 - \mu_0| < \epsilon$ and $|\mu_2 - \mu_0| < \epsilon$,

$$\left|\log p_{\mu_1}(x) - \log p_{\mu_2}(x)\right| \leq \dot{\ell}(x)|\mu_1 - \mu_2|,$$

for some measurable function $\dot{\ell} : \mathbb{R} \to \mathbb{R}$ such that $E_{\mu_0}\dot{\ell}^2 < \infty$.
Hint: for any $\mu_1, \mu_2 \geq \mu > 0$, we have $|\log \mu_1 - \log \mu_2| \leq |\mu_1 - \mu_2|/\mu$.

e. *(5 points)*
State a theorem from the lecture notes and use parts a.–d. to prove that $\sqrt{n}(\mu_n - \mu_0)$ is asymptotically normal under P_{μ_0}. Give the variance of the limit distribution.

The moment estimator for μ_0 is,

$$\tilde{\mu}_n = \frac{\pi}{4}\left(\frac{1}{\overline{X}_n}\right)^2,$$

where $\overline{X}_n$ denotes the sample average.

f. *(5 points)*
Use the delta rule to find the limit distribution for $\sqrt{n}(\tilde{\mu}_n - \mu_0)$. Calculate the relative efficiency of $\hat{\mu}_n$ and $\tilde{\mu}_n$ and explain why you prefer $\hat{\mu}_n$ or $\tilde{\mu}_n$.

Problem 3 *(Variance stabilization)*

Let $X_1, X_2, \ldots$ be *i.i.d.* non-negative real-valued random variables with single-observation distribution Exp(λ) for some $\lambda > 0$, with Lebesgue density $p_\lambda : \mathbb{R} \to [0, \infty)$ where $p_\lambda(x) = 0$ for $x < 0$ and $p_\lambda(x) = \lambda\, e^{-\lambda x}$.

a. *(5 points)*
Find the maximum likelihood estimator $\hat{\lambda}_n$ for λ based on the first n sample points $X_1, \ldots, X_n$.

b. *(5 points)*
Use the delta rule to find a sequence a_n and a constant b such that $a_n(\hat{\lambda}_n - b)$ converges in distribution to a tight but non-degenerate limit distribution. Also give the limit law.

c. *(5 points)*
Perform a variance-stabilizing transformation for $\hat{\lambda}_n$ and construct the associated asymptotic confidence intervals for confidence levels $1-\alpha$, $(0 < \alpha < 1)$.