TENTAMEN

Asymptotic Statistics

WEDNESDAY 14 JANUARY 2015, 14:00 – 17:00 UUR

INSTRUCTIONS

Switch off mobile phone and other electronic devices

Use of a calculator will *not* be necessary and is *not* allowed

On every page, write your name, university and studentnumber

Motivate all answers clearly

If you finish the exam early, hand in your work and leave *quietly*

GRADING

Points per question *(Grade = total/10 + 1)*

1*a*.:	10	2*a*.:	5	3*a*.:	10
1*b*.:	5	2*b*.:	5	3*b*.:	10
1*c*.:	5	2*c*.:	5	3*c*.:	5
1*d*.:	5	2*d*.:	5	3*d*.:	5
1*e*.:	5	2*e*.:	5		
		2*f*.:	5		

Good luck!

[DO NOT TURN THIS PAGE UNTIL SO INSTRUCTED]

PROBLEM 1 *(Uniform integrability)*

Let (X_n) be a sequence of random variables and let X be a random variable. Recall that X_n converges to X *in probability* (notation: $X_n \xrightarrow{P} X$), if for every $\epsilon > 0$, $P(|X_n - X| > \epsilon) \to 0$ as $n \to \infty$. For random variables (X_n) and X that are integrable, we define another form of stochastic convergence as follows: X_n converges to X *in expectation* (or *in* $L_1(P)$, notation: $X_n \xrightarrow{L_1} X$), if $E|X_n - X| \to 0$ as $n \to \infty$.

a. *(10 points)*
Show that if $X_n \xrightarrow{L_1} X$, then also $X_n \xrightarrow{P} X$.

b. *(5 points)*
Construct an example of a sequence (X_n) and a limiting random variable X such that $X_n \xrightarrow{P} X$, but *not* $X_n \xrightarrow{L_1} X$.

c. *(5 points)*
Suppose that $X_n \xrightarrow{P} X$ and prove that if there exist constants $M > 0$ and $N \geq 1$ such that for all $n \geq N$, $P(|X_n| \leq M) = 1$, then $X_n \xrightarrow{L_1} X$.

From the above, it is clear that convergence in expectation implies convergence in probability but the converse is not true in general. The converse *does* hold under the sufficient condition of part *c.*. The question arises whether a sharp extra condition exists, *i.e.* a condition that is not just sufficient but also necessary for convergence in expectation (when combined with convergence in probability). We say that the sequence (X_n) is *uniformly integrable*, if

$$\lim_{M \to \infty} \sup_{n \geq 1} E|X_n| 1\{|X_n| > M\} = 0.$$

d. *(5 points)*
Show that if (X_n) is uniformly integrable and $X_n \xrightarrow{P} X$, then $X_n \xrightarrow{L_1} X$.

e. *(5 points)*
Show that if $X_n \xrightarrow{L_1} X$, then (X_n) is uniformly integrable.

The above constitutes a proof that $X_n \xrightarrow{L_1} X$, *if and only if* $X_n \xrightarrow{P} X$ and (X_n) is uniformly integrable.

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PROBLEM 2 *(Log-normal distributions)*

Let $X_1, X_2, \ldots$ be an *i.i.d.* sample from a so-called *log-normal distribution* with location $\mu_0 \in \mathbb{R}$, defined by the Lebesgue density,

$$p_\mu(x) = (2\pi)^{-1/2}\, x^{-1}\, \exp\bigl(-\tfrac{1}{2}(\log x - \mu)^2\bigr),$$

for $x > 0$. Note that if X has a log-normal distribution with location μ, then $Z := \log X$ is normally distributted with location μ (and variance 1).

a. *(5 points)*
For given $n \geq 1$, find the maximum likelihood estimator $\hat{\mu}_n$ for μ, based on the first n observations.

b. *(5 points)*
Show that $\hat{\mu}_n$ is a consistent estimator sequence for μ_0.

c. *(5 points)*
Calculate the Fisher information I_μ (for a single observation) and show that it is continuous and non-zero as function of $\mu \in \mathbb{R}$.

In what follows, you may need theorem 4.21 from the lecture notes, which is reproduced below problem 3 for your convenience.

d. *(5 points)*
Show that for every μ, the log-density p_μ satisfies $|\log p_{\mu_1}(x) - \log p_{\mu_2}(x)| \leq \dot{\ell}(x)|\mu_1 - \mu_2|$ (for any μ_1, μ_2 in an open neighbourhood of μ), for a measurable $\dot{\ell}(x)$ that satisfies $P_\mu \dot{\ell}^2 < \infty$.

e. *(5 points)*
Derive the limit distribution for $n^{1/2}(\hat{\mu}_n - \mu_0)$. What are the parameters of the limit distribution?

f. *(5 points)*
Let $\tilde{\mu}_n$ be an estimator sequence that is asymptotically normal. What can be said of the relative efficiency of $\hat{\mu}_n$ and $\tilde{\mu}_n$?

[PLEASE CONTINUE ON NEXT PAGE]

PROBLEM 3 *(Domain boundary estimation)*

Let $Y_1, Y_2, \ldots$ be an *i.i.d.* sample from the uniform distribution P_θ on $[0, \theta]$, for some $\theta > 0$. Denote the maximum of the first n observations by $Y_{(n)} = \max\{Y_1, \ldots, Y_n\}$.

a. *(10 points)*
Given $n \geq 1$, find the maximum-likelihood estimator $\hat{\theta}_n$ for θ, based on first n observations.

b. *(10 points)*
Given $n \geq 1$ and θ, find the distribution of $\hat{\theta}_n$.

c. *(5 points)*
Use your answer under *b.* to show that $\hat{\theta}_n$ is consistent for estimation of θ.

Given any estimators $\tilde{\theta}_n$ for the parameter θ, define the *bias* Δ_n of $\tilde{\theta}_n$ by the (θ-dependent) expectation $\Delta_n = P_\theta^n(\tilde{\theta}_n - \theta)$.

d. *(5 points)*
For every $n \geq 1$, give the bias Δ_n of $\hat{\theta}_n$. Find a real-valued sequence (a_n) such that the bias of the estimators $a_n \hat{\theta}_n$ is exactly zero.

Theorem 4.21
For each θ in an open subset of $\mathbb{R}$, let $x \mapsto p_\theta(x)$ be a probability density such that $\theta \mapsto \log p_\theta(x)$ is continuously differentiable for every x and such that, for every θ_1 and θ_2 in a neighbourhood of θ_0,

$$\left|\log p_{\theta_1}(x) - \log p_{\theta_2}(x)\right| \leq \dot{\ell}(x)\,\left|\theta_1 - \theta_2\right|,$$

for a measurable function $\dot{\ell}$ such that $P_{\theta_0}\dot{\ell}^2 < \infty$. Assume that the Fisher information $I_\theta = P_\theta \dot{\ell}_\theta^2$ is continuous and non-zero in θ, for all θ.
Then the maximum-likelihood estimator $\hat{\theta}_n$ based on an i.i.d. sample of size n from p_{θ_0} satisfies that $n^{1/2}(\hat{\theta}_n - \theta_0)$ is asymptotically normal with mean zero and variance $I_{\theta_0}^{-1}$, provided that $\hat{\theta}_n$ is consistent.

[THIS CONCLUDES THE EXAM]