UNIVERSITY OF AMSTERDAM

Faculty of Science

Exam

Asymptotic Statistics
MasterMath SFM (MSc.)

Mid-term exam
Date: Wednesday 26 October 2016
Time: 10.00-12.00

Number of pages: 3 (including front page)
Number of questions: 3
Maximum number of points: 30
For each question is indicated how many points it is worth.

BEFORE YOU START

- Check if your version of the exam is complete.
- Write down **your name, student ID number**, and if applicable the **version number** on **each sheet** that you hand in. Also **number the pages**.
- Your **mobile phone** has to be switched off and be put in your coat or bag. Your **coat and bag** should be on the ground.
- **Tools allowed:** . Other tools are not allowed.

PRACTICAL MATTERS

- The first 30 minutes you are not allowed to leave the room, not even to visit the toilet.
- 15 minutes before the end, you will be warned that the time to hand in is approaching.
- If applicable, fill out the evaluation form at the end of the exam.
- You are obliged to identify yourself at the request of the examiner (or his representative) with a proof of your registration and a valid ID.
- During the examination it is not permitted to visit the toilet, unless the invigilator gives permission to do so.

Good luck!

Problem 1

a. *(5 points)*
Let $X_1, X_2, \ldots$ denote real-valued random variables. Assume convergence of the sums $\sum_{n\geq 1} P(|X_n| > \epsilon) < \infty$ for every $\epsilon > 0$. Show that $X_n \xrightarrow{P} 0$.

b. *(5 points)*
Suppose that the real-valued random variables $X_1, X_2, \ldots$ are *i.i.d.*-$U[0,\theta]$ for some unknown $\theta > 0$. We estimate θ with estimators $T_n(X_1, \ldots, X_n) = \max_{1\leq i\leq n} X_i$. Find a constant $a \in \mathbb{R}$ and a sequence $b_n \to \infty$ such that,

$$b_n(a - T_n) \overset{\theta}{\rightsquigarrow} Z,$$

where Z is distributed tightly but non-degenerately. Which distribution does Z have?

Problem 2 *(Binomials weakly converging to Poisson)*

Let N denote the set of all non-negative integers $\{0, 1, 2, 3, \ldots\}$ and let $X, X_1, X_2, \ldots$ denote random variables taking values in N.

a. *(5 points)*
Show that $X_n \rightsquigarrow X$, if and only if, $P(X_n = x) \to P(X = x)$ for each $x \in N$, as $n \to \infty$.

Recall that for $Y \sim \text{Bin}(m, p)$ and $Z \sim \text{Poisson}(\lambda)$,

$$P(Y = x) = \binom{m}{x} p^x (1-p)^{m-x}, \qquad P(Z = x) = \frac{e^{-\lambda}\lambda^x}{x!},$$

give the densities with respect to the counting measure on N.

b. *(5 points)*
Assume that $X_n \sim \text{Bin}(n, p_n)$ with $p_n \in [0,1]$ for all $n \geq 1$. Show that if, for some constant $\lambda > 0$, $n\,p_n \to \lambda$ as $n \to \infty$, then the sequence (X_n) converges weakly to $\text{Poisson}(\lambda)$.

Hint: *In your calculation of $P(X_n = x)$, use Stirling's approximation for the factorials $n!$ and $(n-x)!$:*

$$\frac{k!}{\sqrt{2\pi k}}\left(\frac{k}{e}\right)^{-k} \to 1,$$

as $k \to \infty$.

Problem 3 *(Asymptotic confidence ellipsoids)*

Let $X_1, X_2, \ldots$ be an *i.i.d.* sample, modelled with a parametric family of distributions P_θ, $\theta \in \mathbb{R}^k$. Assume we have an estimator sequence $T_n(X_1, \ldots, X_n)$ such that for all $\theta \in \mathbb{R}^k$,

$$\sqrt{n}(T_n - \theta) \overset{P_\theta}{\rightsquigarrow} \mathrm{N}_k(0, \Sigma_\theta),$$

for some non-singular $k \times k$ covariance matrix Σ_θ. We also assume that there exists a sequence $S_n(X_1, \ldots, X_n)$ that estimates the covariance martix consistently, that is:

$$S_n \xrightarrow{P_\theta} \Sigma_\theta,$$

for all $\theta \in \mathbb{R}^k$.

a. *(10 points)*
Show that for all $\alpha \in (0, 1)$, the ellipsoids,

$$C_n(X_1, \ldots, X_n) = \left\{\theta \in \mathbb{R}^k \,:\, n(T_n - \theta)^T \, S_n^{-1} \, (T_n - \theta) \leq \chi^2_{k,\alpha}\right\}$$

in $\mathbb{R}^k$ are confidence sets of asymptotic confidence level $1 - \alpha$.