WISB359 Modelling with ODEs and PDEs

Midterm Exam

7 December 2016, 9:00-10:45am

For this midterm exam you are allowed to bring an A4 with notes on one side. The exam will be graded on a scale of 20 points and is worth 40% of your score for the course.

1. A model for the population of Aphids and Ladybugs in the greenhouse of an organic Rose Grower can be described in kinetic notation as follows:

$$
\begin{aligned}
A &\xrightarrow{r} 2A \\
A + A &\xrightarrow{s} A \\
A + L &\xrightarrow{c} (1+\epsilon)L \\
L &\xrightarrow{\delta} \emptyset
\end{aligned}
$$

Here, r is the reproduction rate of Aphids, $s = r/A_0$ is the saturation parameter of the greenhouse, c is the feeding rate of the Ladybugs, ϵ is the consumption efficiency of Ladybugs, and δ is the starvation rate of unfed Ladybugs. All parameters assumed positive.

(a) [1pt] Write down the associated system of differential equations for this model. Make sure you substitute the definition of s given above.

(b) [2pt] Non-dimensionalize these equations, using $A(t) = A_0 a(t)$ and $L(t) = L_0 \ell(t)$ for the Aphid and Ladybug populations, and rescaling time using $t = T\tau$. Choose the scaling constants L_0 and T such that the only non-dimensional parameters left in the equations are the growth-to-decay rate $\beta = r/\delta$ and the critical population ratio $\gamma = \epsilon A_0/L_0$. (*Note: A_0 is not a 'free parameter' here since it is included in the problem desciption. Therefore you are not free to define it in terms of other parameters.*)

(c) [1pt] Suppose there are Aphids but no Ladybugs. Determine the equilibria of the Aphid population and their stability.

$a(0) = 1$

$\ell(0) = 1$

(d) [2pt] Next determine the equilibria in the case where both Aphids and Ladybugs are present. An equilibrium is only relevant from the ecological point of view if both populations are nonnegative. Under what conditions are the equilibria you found relevant?

(e) [4pt] Determine the stability of these equilibria. Under what condition on the original parameters can the Rose Grower expect the aphid population to be driven to extinction? Under what condition will the Aphids and Ladybugs reach a stationary state of coexistence?

2. Consider the following two-point boundary value problem

$$\varepsilon^3 \frac{d^2y}{dx^2} + \left(\frac{\varepsilon}{1-x}\right)\frac{dy}{dx} + y^3 = x, \qquad y(0) = y(1) = 1, \quad \varepsilon > 0.$$

Construct an approximate solution as follows:

(a) [2pt] Derive the one-term outer expansion and show this can satisfy one of the boundary conditions.

(b) [5pt] Derive the one-term inner expansion and show this can satisfy the other boundary condition.

(c) [2pt] Use the matching condition to determine any free constants.

(d) [1pt] Give the composite solution.

$$
\begin{aligned}
f(x) &= f(0) + xf'(0) + \frac{1}{2}x^2 f''(0) + \frac{1}{3!}x^3 f'''(0) + \cdots \\
(a+x)^\gamma &= a^\gamma + \gamma x a^{\gamma-1} + \frac{1}{2}\gamma(\gamma-1)x^2 a^{\gamma-2} + \frac{1}{3!}\gamma(\gamma-1)(\gamma-2)x^3 a^{\gamma-3} + \cdots \\
\frac{1}{1-x} &= 1 + x + x^2 + x^3 + \cdots \\
\frac{1}{(1-x)^2} &= 1 + 2x + 3x^2 + 4x^3 + \cdots \\
\frac{1}{\sqrt{1-x}} &= 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \cdots \\
e^x &= 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots \\
a^x &= e^{x\ln(a)} = 1 + x\ln(a) + \frac{1}{2}(x\ln(a))^2 + \frac{1}{3!}(x\ln(a))^3 + \cdots \\
\sin(x) &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots \\
\cos(x) &= 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 + \cdots \\
\sin(a+x) &= \sin(a) + x\cos(a) - \frac{1}{2}x^2\sin(a) + \cdots \\
\cos(a+x) &= \cos(a) - x\sin(a) - \frac{1}{2}x^2\cos(a) + \cdots \\
\ln(1+x) &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots \\
\ln(a+x) &= \ln(a) + \ln(1+x/a) = \ln(a) + \frac{x}{a} - \frac{1}{2}\left(\frac{x}{a}\right)^2 + \frac{1}{3}\left(\frac{x}{a}\right)^3 + \cdots
\end{aligned}
$$

Table 2.1 Taylor series expansions, about $x = 0$, for some of the more commonly used functions.