Grading Scheme for midterm - examination

Exercise 1: 4 points

Part	1.	2.	3.
points	$1\frac{1}{2}$	$1\frac{1}{2}$	1

Exercise 2. 6 points

Part	1.	2.	3.	4.
points	2	1	$1\frac{1}{2}$	$1\frac{1}{2}$

Solution to Exercise 1)

1. $\times : C^\infty_W(\mathbb{R},\mathbb{R}) \times C^\infty_W(\mathbb{R},\mathbb{R}) \longrightarrow C^\infty_W(\mathbb{R}^2,\mathbb{R}^2)$ is continuous

$$(f,g) \longmapsto (f\times g)(x,y) := (f(x),g(y))$$

Proof:

- we first consider $\times : C^r_W(\mathbb{R},\mathbb{R}) \times C^r_W(\mathbb{R},\mathbb{R}) \longrightarrow C^r_W(\mathbb{R}^2,\mathbb{R}^2)$, $r\geq 0$

  take $f\times g \in C^r_W(\mathbb{R}^2,\mathbb{R}^2)$ and $\mathcal{U}$ open nghd of $f\times g$

  w.l.o.g $\mathcal{U} = \mathcal{N}^r(f\times g, K, \varepsilon)$, $K\subset\mathbb{R}^2$ compact, $\varepsilon>0$

  $K\subset K_1\times K_2$ for $K_1\subset\mathbb{R}$, $K_2\subset\mathbb{R}$ compact

  set $\varepsilon_1 = \varepsilon/2$, $\varepsilon_2 = \varepsilon/2$

  claim: $\times$ maps $\mathcal{N}^r(f, K_1,\varepsilon_1)\times\mathcal{N}^r(f, K_2,\varepsilon_2)$ into $\mathcal{U}$

  Proof: pick $\tilde f\times\tilde g$ with $(\tilde f,\tilde g)\in\mathcal{N}^r(f,K_1,\varepsilon_1)\times\mathcal{N}^r(g,K_2,\varepsilon_2)$

  i.e. $\|D^l\tilde f - D^l f\|_{K_1} < \varepsilon_1$, $\|D^l\tilde g - D^l g\|_{K_2} < \varepsilon_2$ $\forall l = 0,..,r$

  hence: $\|D^l(\tilde f\times\tilde g) - D^l(f\times g)\|_K \leq \|D^l(\tilde f\times\tilde g) - D^l(f\times g)\|_{K_1\times K_2}$

  $$\leq \|D^l(\tilde f\times\tilde g) - D^l(\tilde f\times g)\|_{K_1\times K_2} + \|D^l(\tilde f\times g) - D^l(f\times g)\|_{K_1\times K_2}$$

  $$= \|D^l(0\times(\tilde g - g))\|_{K_1\times K_2} + \|D^l((\tilde f - f)\times 0\|_{K_1\times K_2}$$

  $$= \|D^l(\tilde g) - D^l(g)\|_{K_2} + \|D^l(\tilde f) - D^l(f)\|_{K_1} < \varepsilon_1+\varepsilon_2 = \varepsilon$$ $\square$

$\Longrightarrow$ $\forall r\geq 0$: $\times : C^r_W(\mathbb{R},\mathbb{R})\times C^r_W(\mathbb{R},\mathbb{R}) \to C^r_W(\mathbb{R}^2,\mathbb{R}^2)$ is continuous

- consider $C^\infty_W(\mathbb{R},\mathbb{R})\times C^\infty_W(\mathbb{R},\mathbb{R}) \xrightarrow{\times} C^\infty_W(\mathbb{R},\mathbb{R})$

  this is continuous $\Longleftrightarrow$ $\forall r\geq 0$, the map

$$\begin{tikzcd}
C^\infty_W(\mathbb{R},\mathbb{R}) \times C^\infty_W(\mathbb{R},\mathbb{R}) \arrow[r] \arrow[d, "\text{continuous}"] & C^\infty_W(\mathbb{R},\mathbb{R}) \arrow[r, hook] & C^r_W(\mathbb{R},\mathbb{R}) \\
C^\infty_W(\mathbb{R},\mathbb{R}) \times C^r_W(\mathbb{R},\mathbb{R}) \arrow[urr, "\times"', "\text{continuous}"] & &
\end{tikzcd}$$

is continuous ✓

$\Longrightarrow \quad \times : C^\infty_W(\mathbb{R},\mathbb{R}) \times C^\infty_W(\mathbb{R},\mathbb{R}) \longrightarrow C^\infty_W(\mathbb{R}^2,\mathbb{R}^2)$ is continuous

2. $C^\infty_W(\mathbb{R},\mathbb{R})$ is a topological group, i.e.

(a) $$\begin{array}{ccc} C^\infty_W(\mathbb{R},\mathbb{R}) \times C^\infty_W(\mathbb{R},\mathbb{R}) & \xrightarrow{+} & C^\infty_W(\mathbb{R},\mathbb{R}) \\ (f,g) & \longmapsto & f+g \end{array}$$ is continuous &

(b) $$\begin{array}{ccc} C^\infty_W(\mathbb{R},\mathbb{R}) & \xrightarrow{-} & C^\infty_W(\mathbb{R},\mathbb{R}) \\ f & \longmapsto & -f \end{array}$$ is continuous

concerning (a):

$$\begin{tikzcd}
C^\infty_W(\mathbb{R},\mathbb{R}) \times C^\infty_W(\mathbb{R},\mathbb{R}) \arrow[r, "+"] \arrow[d, "\times"'] & C^\infty_W(\mathbb{R},\mathbb{R}) \\
C^\infty_W(\mathbb{R}^2,\mathbb{R}^2) \arrow[r, "(+)_*"] & C^\infty_W(\mathbb{R}^2,\mathbb{R}) \arrow[u, "(\Delta)^*"']
\end{tikzcd}$$

where $\begin{array}{l} + : \mathbb{R}^2 \longrightarrow \mathbb{R} \\ (x,y) \longmapsto x+y \end{array}$, $\quad \begin{array}{ccc} (+)_* : C^\infty_W(\mathbb{R}^2,\mathbb{R}^2) & \longrightarrow & C^\infty_W(\mathbb{R}^2,\mathbb{R}) \\ h & \longmapsto & + \circ h \end{array}$

and $\begin{array}{l} \Delta : \mathbb{R} \longrightarrow \mathbb{R}^2 \\ x \longmapsto (x,x) \end{array}$, $\quad \begin{array}{ccc} (\Delta)^* : C^\infty_W(\mathbb{R}^2,\mathbb{R}) & \longrightarrow & C^\infty_W(\mathbb{R},\mathbb{R}) \\ h & \longmapsto & h \circ \Delta \end{array}$

we saw in the Exercises that pre- and postcomposition induces continuous maps w.r.t the weak topology

$\Longrightarrow$ $+$ can be written as the composition of continuous maps and hence is continuous

concering (b) $\mathcal{C}^\infty_W(\mathbb{R},\mathbb{R}) \xrightarrow{\;-\;} \mathcal{C}^\infty_W(\mathbb{R},\mathbb{R})$,
$$f \longmapsto \bar{f}$$

on can write this as $(-1)_*$ where $(-1) : \mathbb{R} \to \mathbb{R}$
$$x \mapsto -x$$

hence $-$ is continuous

3. $\mathbb{R} \times \mathcal{C}^\infty_S(\mathbb{R},\mathbb{R}) \xrightarrow{\;\cdot\;} \mathcal{C}^\infty_S(\mathbb{R},\mathbb{R})$ is not continuous

<u>Proof</u>: if it were, it would map convergent sequences to convergent sequences

consider
$$\begin{array}{ccc} (\frac{1}{n}, \underbrace{\mathbb{1}}) & \longrightarrow & (0, \mathbb{1}) \\ \cdot \big\downarrow \; \mathbb{1}(x) = 1 & & \big\downarrow \cdot \\ \frac{1}{n}\mathbb{1} & \not\longrightarrow & 0 \end{array}$$

we saw in the exercises that $(\frac{1}{n}\mathbb{1})_{n\geq 1}$ does not converge to 0 in the strong topology $\Rightarrow$ contradiction

Solution to Exercise 2)

1. $W : \mathrm{Imm}^\infty_S(S^1, \mathbb{R}^2) \longrightarrow C^\infty_S(S^1, S^1)$ is continuous

$$u \longmapsto W_u = \frac{du/d\theta}{\|du/d\theta\|}$$

Proof: write W as: $\mathrm{Imm}^\infty_S(S^1, \mathbb{R}^2) \longrightarrow C^\infty_S(S^1, \mathbb{R}^2\setminus\{0\}) \xrightarrow{(p)_*} C^\infty_S(S^1, S^1)$

$$u \longmapsto \frac{du}{d\theta}$$

where $p : \mathbb{R}^2\setminus\{0\} \longrightarrow S^1$

$$x \longmapsto \frac{x}{\|x\|}$$

remains to show: $\mathrm{Imm}^\infty_S(S^1, \mathbb{R}^2) \xrightarrow{\frac{d}{d\theta}} C^\infty_S(S^1, \mathbb{R}^2\setminus\{0\})$ is continuous

$$\Updownarrow$$

$$\begin{tikzcd}
\forall r \geq 0 : \mathrm{Imm}^\infty_S(S^1, \mathbb{R}^2) \arrow[r, "\frac{d}{d\theta}"] \arrow[d] & C^\infty_S(S^1, \mathbb{R}^2\setminus\{0\}) \arrow[r, hook] & C^r_S(S^1, \mathbb{R}^2\setminus\{0\}) \\
\mathrm{Imm}^{r+1}_S(S^1, \mathbb{R}^2) \arrow[urr, "\frac{d}{d\theta}"'] & &
\end{tikzcd}$$

is continuous

pick open $\mathcal{U}$ of $\frac{du}{d\theta} \in C^r_S(S^1, \mathbb{R}^2\setminus\{0\})$ for $u \in \mathrm{Imm}^{r+1}_S(S^1, \mathbb{R}^2)$

w.l.o.g. $\mathcal{U} = \mathcal{N}^r\left(\frac{du}{d\theta}, (U, \varphi), \mathbb{R}^2\setminus\{0\}, K, \varepsilon\right)$

claim: $\exists\, \mathcal{V} = \mathcal{N}^{r+1}(u, (U, \varphi), \mathbb{R}, K, \tilde{\varepsilon})$ which gets mapped into $\mathcal{U}$ under $\frac{d}{d\theta}$

since $\mathrm{Imm}^{r+1}_S(S^1, \mathbb{R}^2)$ is open, we can assume $\mathcal{V} \subset \mathrm{Imm}^{r+1}_S(S^1, \mathbb{R}^2)$

$v \in \mathcal{V} \Leftrightarrow \|D^\ell(v \circ \varphi^{-1}) - D^\ell(u \circ \varphi^{-1})\|_{\varphi(K)} < \tilde{\varepsilon} \quad \forall\, \ell = 0, \dots, r+1$

Consider $\| D^{\ell}(\frac{dv}{d\theta} \circ \varphi^{-1}) - D^{\ell}(\frac{du}{d\theta} \circ \varphi^{-1}) \|_{\varphi(K)} =$

$$\| D^{\ell+1}(v \circ \varphi^{-1})(T\varphi(\tfrac{\partial}{\partial\theta}) \circ \varphi^{-1}, -) - D^{\ell+1}(u \circ \varphi^{-1})(T\varphi(\tfrac{\partial}{\partial\theta}) \circ \varphi^{-1}, -) \|_{\varphi(K)}$$

$$\leq \| T\varphi(\tfrac{\partial}{\partial\theta}) \|_{\varphi(K)} \, \| D^{\ell+1}(v \circ \varphi^{-1}) - D^{\ell+1}(u \circ \varphi^{-1}) \|_{\varphi(K)}$$

$$\leq \| T\varphi(\tfrac{\partial}{\partial\theta}) \|_{\varphi(K)} \, \tilde{\varepsilon} < \varepsilon$$

if we choose $0 < \tilde{\varepsilon} < \dfrac{1}{1+\max(\| T\varphi(\frac{\partial}{\partial\theta}) \|_{\varphi(K)}, \|_{\varphi^{-1}}}$ $\square$

2. $u_\infty(\theta) = (\cos\theta, \sin 2\theta)$

(a) u_∞ is an immersion:

$$du_\infty(\theta) = (-\sin\theta, 2\cos 2\theta) \overset{!}{=} 0 \iff$$

$$\sin\theta = 0, \; \cos 2\theta = 0$$

$$\Updownarrow \qquad\qquad \not\Downarrow$$

$$\theta = 0, \pi, \cancel{2\pi} \implies \cos 0 = 1, \; \cos 2\pi = 1, \; \cancel{\cos 4\pi = 1}$$

$\implies du_\infty \neq 0$, hence u_∞ is an immersion

(b) $(0,-1) \notin$ image of W_{u_∞}

$$(0,-1) \overset{!}{=} W_{u_\infty}(\theta) \iff du_\infty(\theta) = (0, -\lambda) \quad \text{for } \lambda > 0$$

$$\Downarrow$$

$$\cancel{\iff} \quad \theta = 0, \pi, \cancel{2\pi} \implies$$

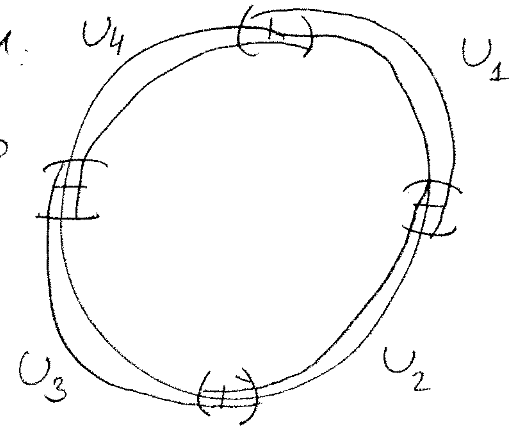
$$du_\infty(0) = (0,2), \; du_\infty(\pi) = (0,2) \quad \lightning$$

$\square$

3. find $\mathcal{U}$ open nghd of $f \in \mathcal{C}^\infty_S(S^1,S^1)$ such that

$$\text{if } g \in \mathcal{U} \Longrightarrow g(x) \neq -f(x) \quad \forall x \in S^1$$

consider open cover of S^1:



important: $U_i \cap -U_i = \phi$

$\forall i = 1,2,3,4$

$\forall x \in S^1 \ \exists (V_x, \varphi_x)$ chart around ~~open nghd of~~ x s.t. $f(V_x) \subset U_{i(x)}$

Since S^1 is compact, we can choose a finite subcover $V_1, \dots, V_N$

& compacts $L_1 \subset V_1, \dots, L_N \subset V_N$ which still cover S^1

consider $\mathcal{N}^D(f, (V_i, \varphi_i), U_i, L_i, \varepsilon)$

$$g \in \mathcal{N}^D(f, (V_i, \varphi_i), U_i, L_i, \varepsilon) \Longrightarrow$$

$$x \in L_i \Longrightarrow g(x) \in U_i \ni f(x) \xRightarrow{U_i \cap -U_i = \phi} g(x) \neq -f(x)$$

□

4. $Emb^\infty(S^1, \mathbb{R}^2) \subset \mathcal{C}^\infty_S(S^1, \mathbb{R}^2)$ is not dense

Proof: pick $u_\infty \in \mathcal{C}^\infty_S(S^1, \mathbb{R}^2)$

choose an open nghd $\mathcal{U}$ of u_∞ s.t.

(1) $\mathcal{U} \subset Imm^\infty_S(S^1, \mathbb{R}^2)$

(2) W maps $\mathcal{U}$ into $\mathcal{V} \subset \mathcal{C}^\infty_S(S^1, S^1)$

s.t. $g \in \mathcal{V} \Rightarrow g(x) \neq -W_{u_\infty}(x) \quad \forall x \in S^1$

$\Longrightarrow g$ & W_{u_∞} are homotopic by 2,(b) W_{u_∞} is not surjective, hence homotopic to a constant map & so is any $g \in \mathcal{V}$

<u>claim</u>: $\mathcal{U}$ does not contain any embedding

<u>Proof</u>: $u \in \mathcal{U}$ embedding $\Rightarrow$ W_u is homotopic to $\pm id$
but $W_u \in \mathcal{V}$ $\Rightarrow$ W_u is homotopic to a constant map

↯ since this would imply that $\pm id$ is homotopic to the constant map □