Differential Topology - Examination (January 23rd, 2013)

1. *Please...*
   - *(a) make sure your name and student number are written on every sheet of paper that you hand in;*
   - *(b) start each exercise on a new sheet of paper and number each sheet.*
2. *All results from the lectures and the exercises can be taken for granted, but must be stated when used.*

Exercise 1 (3 points)**.** Let P, M and N be smooth manifolds. Moreover, let

$$\hat{F} : P \times M \to N$$

be a smooth map. We define

$$F : P \to \mathcal{C}^\infty(M, N), \quad p \mapsto F_p(m) := \hat{F}(p, m).$$

1. Show that F is in general *not* continuous if one equips $\mathcal{C}^\infty(M, N)$ with the strong $\mathcal{C}^\infty$-topology.
2. Prove that F is continuous if one equips $\mathcal{C}^\infty(M, N)$ with the weak $\mathcal{C}^\infty$-topology.

Exercise 2 (4 points)**.** Let M be a manifold of dimension $m \geq 2$ and N a manifold of dimension $2m - 1$.

1. Prove that the subset

   $$Y := \{f : M \to N \text{ smooth} : \forall x \in M \ \text{rank}(d_x f) \text{ is either } m \text{ or } m-1\} \quad \subset \mathcal{C}^\infty(M, N)$$

   is residual.

2. Prove that there is a residual subset $Z \subset \mathcal{C}^\infty(M, N)$ such that if $f \in Z$,

   $$X_{m-1}(f) := \{x \in M : d_x f \text{ has rank } m - 1\} \subset M$$

   is a submanifold of dimension 0, which is closed.

Exercise 3 (3 points)**.** Let v be a vector field on the n-dimensional disk

$$D^n := \{x \in \mathbb{R}^n : ||x|| \leq 1\} \quad \subset \mathbb{R}^n,$$

which does not vanish on the boundary S^{n-1}.

1. Prove that if v has no zeros on D^n, then the map

   $$\phi_v : S^{n-1} \to S^{n-1}, \qquad x \mapsto \frac{v(x)}{||v(x)||}$$

   has degree 0.

2. Suppose that v is transverse to the boundary, i.e.

   $$T_x S^{n-1} + < v(x) >= T_x \mathbb{R}^n$$

   holds for all $x \in S^{n-1}$. Show that such a vector field v must have a zero in the interior of D^n.
   (Hint: It might help to consider the decomposition of $v|_{S^{n-1}} = v_{||} + v_\perp$, where $v_{||}$ is tangential to S^{n-1}, while $v_\perp$ is perpendicular to S^{n-1}.)