Grading scheme for examination

Exercise 1: 3 points

Part	1.	2.
Points	1	2

Exercise 2: 4 points

Part	1.	2.
Points	2	2

Exercise 3: 3 points

Part	1.	2.
Points	1	2

Solution to Exercise 1)

$\hat{F} : P \times M \xrightarrow{C^\infty} N$, def $F : P \longrightarrow C^\infty(M,N)$

$p \longmapsto F_p(m) = \hat{F}(p,m)$

1. claim: F is in general not continuous w.r.t. the strong C^∞-topology on $C^\infty(M,N)$

Proof: consider $\hat{F} : \mathbb{R} \times \mathbb{R} \xrightarrow{C^\infty} \mathbb{R}$

$(x,y) \longmapsto x$

$F : \mathbb{R} \longrightarrow C^\infty(\mathbb{R},\mathbb{R})$

$x \longmapsto (y \mapsto x) =: c_x$

we saw in the Exercises that F is NOT continuous w.r.t. the strong C^∞-topology

(for instance consider $x_n = \frac{1}{n} \longrightarrow x_\infty = 0$

but $c_{x_n} \not\longrightarrow c_0$ w.r.t strong top.

$\square$

2. claim: F is continuous w.r.t. the weak C^∞-topology

Proof: this can be directly verified

alternatively: composition is continuous w.r.t. the weak C^∞-topology

given $p \in P \rightsquigarrow \Gamma_p : M \xrightarrow{C^\infty} P \times M$

$m \longmapsto (p,m)$

$\hat{F}$ can be written as

$$
\begin{tikzcd}
P \arrow[rr, "\hat{F}"] \arrow[dr, "\Gamma"'] & & \mathcal{C}^\infty(N,M) \\
& \mathcal{C}^\infty(M, P\times M) \arrow[ur, "F^*"'] &
\end{tikzcd}
$$

continuous w.r.t weak $\mathcal{C}^\infty$-topology

$\mathcal{C}^\infty(M, P\times M)$ is homeomorphic $\mathcal{C}^\infty(M,P) \times \mathcal{C}^\infty(M,M)$

$$
\begin{tikzcd}
p \arrow[rr, mapsto] & & (m \mapsto p) \\
P \arrow[r] & \mathcal{C}^\infty(M,P)\times\mathcal{C}^\infty(M,M) \arrow[r] \arrow[dr] & \mathcal{C}^\infty(M,P) \\
p \arrow[rr, mapsto, "\text{continuous}"'] & & \mathcal{C}^\infty(M,M) \\
& & id
\end{tikzcd}
$$

continuous w.r.t weak $\underbrace{\mathcal{C}^\infty\text{-topology}}_{\text{see below}}$

$$
\begin{aligned}
P &\longrightarrow \mathcal{C}^\infty(M,P) \\
p &\longmapsto (m \mapsto p)
\end{aligned}
$$

can be considered as $\underbrace{\mathcal{C}^\infty(M,\{*\}) \times \mathcal{C}^\infty(\{*\},P) \longrightarrow}_{\text{continuous}} \mathcal{C}^\infty(M$

$\square$

Solutions to Exercise 2)

$\begin{matrix} M \\ N \end{matrix}$ mfds of dim $\begin{matrix} m \\ 2m-1 \end{matrix}$ $(m \geq 2)$

1. claim: $\mathcal{Y} := \{f : M \xrightarrow{\mathcal{C}^\infty} N : \forall x \in M \ \text{rank}\, d_x f = m \text{ or } m-1\}$

$\cap$

$\mathcal{C}^\infty(M, N)$ is residual

Proof: $\text{rank}\, d_x f \in \{0, .., m\}$ $\Longrightarrow$

$$\mathcal{Y} = \{f : M \xrightarrow{\mathcal{C}^\infty} N : \forall x \in M \ \text{rank}(d_x f) \notin \{0, .., m-2\}\}$$

recall: $j^1 f : M \longrightarrow J^1(M, N)$ first jet prolongation of f

$A_k \subset J^1(M, N)$ ~~A~~ is a submfd of dim $= \dim M + \dim N$
$\quad \Downarrow$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + k(\dim M + \dim N - k)$
$\xi : T_x M \longrightarrow T_y N$ of rank $= k$

$$\mathcal{Y} = \{f : M \xrightarrow{\mathcal{C}^\infty} N : \text{im}(j^1 f) \cap A_k = \emptyset \text{ for } k = 0, .., m-2\}$$

claim: $j^1 f \pitchfork A_k \Longrightarrow \text{im}(j^1 f) \cap A_k = \emptyset$ for $k = 0, .., m-2$

Proof: if $\dim M + \dim A_k \overset{(*)}{<} \dim J^1(M, N)$,

no intersection point of ~~j~~ $\text{im}(j^1 f)$ and A_k

can be transversal $\Longrightarrow$

have to prove $(*)$ for $k = 0, .., m-2$:

$$\cancel{m} + m + \cancel{2m-1} + k(m + 2m - 1 - k) \overset{?}{<} \cancel{m} + \cancel{2m-1} + m(2m-1)$$

$$\Updownarrow$$

$$m + k(3m - 1 - k) \overset{?}{<} 2m^2 - m$$

$$\Updownarrow$$

$$m + (m-2)(\underbrace{3m-1-m+2}_{2m+1}) < 2m^2-2$$

$$\Updownarrow$$

$$m + \cancel{2m^2} + m - 4m - \cancel{2} < \cancel{2m^2} - \cancel{2}$$

$$\Updownarrow$$

$$-2m < 0 \quad \checkmark$$

$\square$

<u>hence</u>: $\mathcal{Y} = \{ f : M \xrightarrow{C^\infty} N : j^1 f \pitchfork A_k \text{ for } k = 0, \ldots, m-2 \}$

$\Longrightarrow$ transversality theorem implies that

$\mathcal{Y} \in C^\infty(M,N)$ is residual

2. $\exists \; Z \subset C^\infty(M,N)$ residual s.t.

$f \in Z \Rightarrow X_{m-1}(f) = \{ x \in M : d_x f \text{ has rank } m-1 \} \subset M$

is a submfd of dim 0, which is closed

<u>Proof</u>: def: $Z = \{ f : M \xrightarrow{C^\infty} N : j^1 f \pitchfork A_k \text{ for } k = 0, \ldots, m-1 \}$

$= \mathcal{Y} \cap \{ f : M \xrightarrow{C^\infty} N : j^1 f \pitchfork A_{m-1} \}$

residual by transversality

$(j^1 f)^{-1}(A_{m-1}) \subset M$ is a submfd of codim = codim of A_{m-1}

$\quad \| $

$X_{m-1}(f)$

codim of $A_{m-1} = \cancel{m} + 2\cancel{m} - 1 + m(2m-1)$

$\quad - (\cancel{m} + 2\cancel{m} - 1 + (m-1)(\cancel{m} + 2m - \cancel{1} - \cancel{m} + \cancel{1}))$

$= 2m^2 - m - (m-1)(2m) = m$

$\Rightarrow$ codim of $X_{m-1}(f) = m$ $\Rightarrow$ dim of $X_{m-1}(f) = 0$

<u>claim</u>: $X_{m-1}(f)$ is closed

<u>Proof</u> $f \in Z \Rightarrow \operatorname{rank}(d_x f) \in \{m-1, m\}$

$X_m(f) = \{x \in M : \operatorname{rank}(d_x f) = m\}$ is open

$\Longrightarrow X_{m-1}(f) = M \setminus X_m(f)$ is closed $\square$

Solutions to Exercise 3)

1. v vector field on $D^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$,

which does not vanish on boundary S^{n-1}

<u>claim</u>: if v has no zeros on $D^n \Longrightarrow$

$$\phi_v : S^{n-1} \longrightarrow S^{n-1}, \quad x \longmapsto \frac{v(x)}{\|v(x)\|} \quad \text{has deg } 0$$

<u>Proof</u>: consider $H : S^{n-1} \times [0,1] \longrightarrow S^{n-1}$

$$(x, r) \longmapsto \frac{v(xr)}{\|v(xr)\|}$$

homotopy between ϕ_v and constant map

$\Longrightarrow \deg \phi_v = \deg(\text{constant map}) = 0$ $\square$

2. suppose $T_x S^{n-1} + \langle v(x) \rangle = T_x \mathbb{R}^n$

<u>claim</u>: v has a zero in the interior of D^n

<u>Proof</u>: write $v = v_\parallel + v_\perp$ where

$$\langle v_\parallel(x), x \rangle = 0, \quad v_\perp(x) = \lambda x$$

consider $H : S^{n-1} \times [0,1] \longrightarrow S^{n-1}$

$$(x, t) \longmapsto \frac{t v_\parallel + v_\perp}{\|t v_\parallel + v_\perp\|}$$

well-defined: $t v_\parallel(x) + v_\perp(x) = 0 \Rightarrow v_\perp(x) = 0$

this would contradict $T_x S^{n-1} + \langle v(x) \rangle = T_x \mathbb{R}^n$

since $v_\parallel(x) \in T_x S^{n-1}$

$\Rightarrow$ $\exists$ homotopy between ϕ_v & $\phi_{v_\perp}$

now: $\phi_{v_\perp}(x) = \dfrac{\lambda(x) \cdot x}{\|\lambda(x) \cdot x\|} = \pm x$

$$\deg \phi_{v_\perp} = \deg \pm x = \pm 1$$

$$\overset{H}{\Longrightarrow} \quad \deg \phi_v = \deg \phi_{v_\perp} = \pm 1$$

Suppose v has no zero on D^n $\overset{a)}{\Longrightarrow}$ $\deg \phi_v = 0$ $\lightning$ $\square$