Differential Topology - Retake-Examination (Feburary 28th, 2013)

1. *Please...*
   (a) *make sure your name and student number are written on every sheet of paper that you hand in;*
   (b) *start each exercise on a new sheet of paper and number each sheet.*
2. *All results from the lectures and the exercises can be taken for granted, but must be stated when used.*

Exercise 1 (3 points)**.**

1. Consider the restriction map

$$\alpha : \mathcal{C}^1_W(S^1, \mathbb{R}) \to \mathcal{C}^1_W(S^1 \setminus \{(1,0)\}, \mathbb{R}), \quad f \mapsto f|_{S^1\setminus\{(1,0)\}}.$$

   Prove that the image of α is not an open subset of $\mathcal{C}^1_W(S^1 \setminus \{(1,0)\}, \mathbb{R})$.

2. Let $U \subset \mathbb{R}^n$ be an open subsets. Consider the restriction map

$$\beta : \mathcal{C}^1_S(\mathbb{R}^n, \mathbb{R}) \to \mathcal{C}^1_S(U, \mathbb{R}), \quad f \mapsto f|_U.$$

   Prove that the image of β is an open subset of $\mathcal{C}^1_S(U, \mathbb{R})$.

Exercise 2 (4 points)**.** Let M be a compact manifold.

1. Let $f : M \to M$ be a smooth map. A point $x \in M$ is a *fixed point* of f if $f(x) = x$. A fixed point x of f is *Lefschetz* if the differential

$$d_x f : T_x M \to T_x M$$

   of f at x *does not* have $+1$ as an eigenvalue.

   (a) Prove that if all fixed points of f are Lefschetz, then f has only finitely many fixed points.

   (b) Prove that the set of smooth maps $f : M \to M$, all whose fixed points are Lefschetz, is an open and dense subset of $\mathcal{C}^\infty_S(M, M)$.

   (Hint: Consider the map $(\mathrm{id}, f) : M \to M \times M$).

2. Let $f : M \to M$ be a smooth map, all whose fixed points are Lefschetz. We define the *mod 2 Lefschetz number* of f to be

$$L(f) := \left(\sum_{x \text{ fixed point of } f} 1 \right) \bmod 2.$$

   If f and g are homotopic smooth maps from M to M, all whose fixed points are Lefschetz, then $L(f) = L(g)$.

   Prove that any smooth map $f : S^n \to S^n$, $n > 0$, of degree 0 has at least one fixed point.

Exercise 3 (3 points)**.** Recall that a smooth function $f : M \to \mathbb{R}$ is called *Morse* if $df : M \to T^*M$ intersects the zero section

$$Z := \{(x, 0) \in T^*M : x \in M\}$$

transversally.

Let M be a submanifold of $\mathbb{R}^{q+1}$ $(q > 0)$. For each $v \in S^q$, define

$$f_v : M \to \mathbb{R}, \quad x \mapsto < v, x > .$$

Prove that the subset of those $v \in S^q$, for which f_v is a Morse function, is dense in S^q.