INLDS 2016 Examination Problems

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1 Local bifurcations in BT normal form

Consider the following planar system depending on two parameters:

$$\begin{cases} \dot{\xi}_1 &= \xi_2, \\ \dot{\xi}_2 &= \beta_1 + \beta_2 \xi_1 + \xi_1^2 - \xi_1 \xi_2. \end{cases} \tag{1.1}$$

This is Bogdanov's normal form for the codim 2 Bogdanov-Takens bifurcation.

1. Derive explicit formulas for the saddle-node and Hopf bifurcation curves in the parameter plane of (1.1).

2. Obtain a symbolic expression for the quadratic normal form coefficient a along the saddle-node curve when $\beta \neq 0$.

3. Compute symbolically the first Lyapunov coefficient l_1 along the Hopf curve.

4. Verify your results by simulations with `pplane` and by numerical continuation in `MatCont`.

2 Hopf bifurcation in ZH normal form

Consider the planar system

$$\begin{cases} \dot{\xi} &= \beta_1 + \xi^2 + \rho^2, \\ \dot{\rho} &= \rho(\beta_2 + \theta\xi + \xi^2), \end{cases} \tag{2.1}$$

where $\theta < 0$. This system appears in the analysis of the fold-Hopf codim 2 bifurcation of equilibria as an amplitude system for the truncated normal form, so that $\rho \geq 0$.

1. Find parameter values at which bifurcations of equilibria with $\rho = 0$ occur.

2. Verify that Hopf bifurcation of an equilibrium with small $\rho > 0$ happens in the system (2.1) at the line

   $$T = \{(\beta_1, \beta_2) : \beta_2 = 0,\ \beta_1 < 0\}.$$

   Derive an expression for the first Lyapunov coefficient l_1 along the Hopf line T and predict stability of the bifurcating cycle.

3. Illustrate your predictions by simulations with `pplane` and by numerical continuation in `MatCont`.

4. Try to obtain as complete as possible bifurcation diagram of (2.1) near the origin for small $\|\beta\|$.

3 Hopf bifurcation in R2 normal form

Consider the planar system

$$\begin{cases} \dot{\zeta}_1 &= \zeta_2, \\ \dot{\zeta}_2 &= \beta_1\zeta_1 + \beta_2\zeta_2 - \zeta_1^3 - \zeta_1^2\zeta_2. \end{cases} \tag{3.1}$$

This system appears in the study of codim 2 bifurcation of limit cycles corresponding to a double multiplier -1.

1. Verify that Hopf bifurcation of the trivial equilibrium $(\zeta_1, \zeta_2) = (0, 0)$ of (3.1) happens at the line

$$H^{(1)} = \{(\beta_1, \beta_2) : \beta_2 = 0,\ \beta_1 < 0\},$$

2. Verify that Hopf bifurcation of the nontrivial equilibria $(\zeta_1, \zeta_2) \neq (0, 0)$ of (3.1) happens at the line

$$H^{(2)} = \{(\beta_1, \beta_2) : \beta_1 = \beta_2,\ \beta_1 > 0\},$$

3. Compute symbolically the first Lyapunov coefficient l_1 along the lines $H^{(1)}$ and $H^{(2)}$ and predict stability of the bifurcating cycles.

4. Verify your results by simulations with `pplane` and by numerical continuation in `MatCont`.

5. Try to obtain as complete as possible bifurcation diagram of (3.1) near the origin for small $\|\beta\|$.

4 Prey-predator dynamics - I

Consider the following prey-predator model depending on two positive parameters (α, δ)

$$\begin{cases} \dot{x} &= x - \dfrac{xy}{1+\alpha x}, \\ \dot{y} &= -y - \delta y^2 + \dfrac{xy}{1+\alpha x}, \end{cases} \tag{4.1}$$

for $x, y \geq 0$.

1. Derive equations for the saddle-node and Hopf bifurcations of positive equilibria in the system. *Hint*: Consider the orbitally-equivalent to (4.1) polynomial system

$$\begin{cases} \dot{x} &= x(1+\alpha x) - xy, \\ \dot{y} &= -(y+\delta y^2)(1+\alpha x) + xy. \end{cases} \tag{4.2}$$

2. Prove that a Bogdanov-Takens bifurcation occurs in the system (4.2) and find the corresponding parameter values.

3. Compute the coefficients a and b of the BT-normal form.

4. Use `pplane` and `MatCont` to produce representative phase portraits of the model and to sketch its simplest possible bifurcation diagram.

5 Prey-predator dynamics - II

Consder the following prey-predator model depending on two parameters (l, m)

$$\begin{cases} \dot{x} &= x(x-l)(1-x) - xy, \\ \dot{y} &= -y(m-x), \end{cases} \tag{5.1}$$

where $m > 0$ and $0 < l < 1$, and $x, y \geq 0$.

1. Derive equations for the borders of a domain in the (l, m)-plane, in which the model has a positive equilibrium.

2. Derive an equation for the Hopf bifurcation of the positive equilibrium of (5.1). Prove that this bifurcation is supercritical, i.e. gives rise to a stable periodic orbit.

3. Use `pplane` and `MatCont` to produce representative phase portraits of the model and sketch its simplest possible bifurcation diagram. *Hints*: Fix $l = \frac{1}{2}$ and plot the phase portraits for several different values of m.

4. There is a global (heteroclinic) bifurcation in the system. Find numerically m_{Hom} for the heteroclinic parameter value when $l = \frac{1}{2}$.

6 Prey-predator model - III

Study the following prey-predator model depending on two positive parameters (α, β)

$$\begin{cases} \dot{x} &= x - \dfrac{xy}{(1+\alpha x)(1+\beta y)}, \\ \dot{y} &= -y + \dfrac{xy}{(1+\alpha x)(1+\beta y)}, \end{cases} \tag{6.1}$$

by combining analytical and numerical methods. Consider only $x, y \geq 0$.

1. Derive an equation for the saddle-node bifurcation in the system. *Hint*: Introduce the orbitally-equivalent to (6.1) polynomial system

$$\begin{cases} \dot{x} &= x(1+\alpha x)(1+\beta y) - xy \equiv F_1, \\ \dot{y} &= -y(1+\alpha x)(1+\beta y) + xy \equiv F_2. \end{cases} \tag{6.2}$$

2. Verify that for $\alpha = \beta$ with $0 < \beta < \frac{1}{4}$ the system (6.2) has a positive equilibrium with two purely imaginary eigenvalues. Prove that for these parameter values the system has a family of closed periodic orbits surrounding the equilibrium. *Hint*: Consider the transformation $(x, y, t) \to (y, x, -t)$ and conclude that the system is reversible.

3. Prove that system (6.2) has no periodic orbits for all other combinations of parameters. *Hint:* Show that $\text{div}(gF) = (\alpha - \beta)xg$, where $g = x^a y^b$ with some constants a and b. Also use the fact that inside any periodic orbit must be at least one equilibrium point.

4. Use `pplane` to produce representative phase portraits of the model and sketch its bifurcation diagram.

7 Prey-predator model - IV

Consider the following prey-predator model

$$\begin{cases} \dot{v} &= v\left(1 - \dfrac{p}{1+\beta v}\right), \\ \dot{p} &= p\left(-\gamma + \dfrac{v}{1+\beta v}\right), \end{cases} \tag{7.1}$$

where $\beta, \gamma > 0$ and $v, p \geq 0$.

1. Prove that the system has a unique positive equilibrium

$$(v_0, p_0) = \left(\frac{\gamma}{1-\beta\gamma}, \frac{1}{1-\beta\gamma}\right)$$

   when $1 - \beta\gamma > 0$. Show that this equilibrium is unstable.

2. Prove that any orbit of (7.1) starting in the positive quadrant tends to this equilibrium as $t \to -\infty$. You may assume that any such backward orbit is bounded.

   Hints: Consider the orbitally-equivalent to (7.1) in the positive quadrant polynomial system

$$\begin{cases} \dot{v} &= v(1 + \beta v - p), \\ \dot{p} &= p(-\gamma + (1-\beta\gamma)v), \end{cases} \tag{7.2}$$

   then introduce new variables, namely:

$$\begin{cases} x &= \ln v, \\ y &= \ln p. \end{cases}$$

   Apply Bendixson's Criterion and the Poincaré-Bendixson Theorem to the resulting system.

3. Illustrate your results by simulations with `pplane`.

8 Center manifold in Rössler system

Consider the following system

$$\begin{cases} \dot{x} &= -(y+z) \\ \dot{y} &= x + Ay \\ \dot{z} &= Bx + (x - C)z. \end{cases} \tag{8.1}$$

1. Fix B and C and compute the parameter value $A = A_0(B, C)$ at which the system (8.1) has an equilibrium with eigenvalue $\lambda_1 = 0$.

2. Compute the quadratic coefficient $a = a(B, C)$ of the restriction of the system to its one-dimensional center manifold at this bifurcation:

$$\dot{\xi} = a\xi^2 + O(\xi^3).$$

3. What happens to the critical equilibrium for small $|A - A_0| > 0$?

9 Hopf bifurcation in adaptive control - I

Consider the following 3D system

$$\begin{cases} \dot{x} &= \mu x + y, \\ \dot{y} &= -x + \mu y - xz, \\ \dot{z} &= -z + ax^2, \end{cases} \tag{9.1}$$

where $a > 0$. This system appears in the control theory.

1. Verify that system (9.1) exhibits a Hopf bifurcation of the equilibrium $(x, y, z) = (0, 0, 0)$ at the parameter value $\mu = 0$.

2. Compute the corresponding first Lyapunov coefficient l_1 and predict the direction of the Hopf bifurcation and the stability of the bifurcating limit cycle.

3. Verify your predictions by simulations or by numerical continuation of the cycle in `MatCont`.

10 Hopf bifurcation in adaptive control - II

Consider the following 3D system

$$\begin{cases} \dot{x} &= \mu x - y - xz, \\ \dot{y} &= \mu y + x, \\ \dot{z} &= -z + y^2 + x^2 z, \end{cases} \tag{10.1}$$

appearing in the control theory.

1. Verify that the system (10.1) exhibits a Hopf bifurcation of the equilibrium $(x, y, z) = (0, 0, 0)$ at $\mu = 0$.

2. Compute the corresponding first Lyapunov coefficient l_1 and predict the direction of the Hopf bifurcation and the stability of the bifurcating limit cycle.

3. Verify your predictions by simulations or by numerical continuation of the cycle in `MatCont`.

11 Double zero bifurcation in Lorenz system

Consider the famous Lorenz system

$$\begin{cases} \dot{x} = \sigma(y - x), \\ \dot{y} = rx - y - xz, \\ \dot{z} = -bz + xy, \end{cases} \tag{11.1}$$

with any $(\sigma, r, b) \in \mathbb{R}^3$, i.e. without requiring all parameters to be positive[1]. For $b = -2$:

1. Find critical parameters values (σ_0, r_0), such that the trivial equilibrium $O = (0, 0, 0)$ has a double zero eigenvalue.

2. Fix (σ, r) at their critical values and make the substitution

$$\begin{cases} x = u + v, \\ y = u, \\ z = w \end{cases}$$

   Check that the linear part of the resulting system in the (u, v, w)-coordinates has the Jordan normal form. Work further with the transformed system at the critical parameter values.

3. Show that the critical 2D center manifold $W_0^c(O)$ is given by the graph of the function

$$w = -\frac{1}{2}(u + v)^2.$$

4. Analyze the restriction of the system to $W_0^c(O)$. In particular, prove that the restricted system is Hamiltonian and produce its phase portrait in the (u, v)-plane with `pplane`.

5. Which conclusions about the behavior of the 3D system (11.1) with $b = -2$ at (σ_0, r_0) can be made ?

[1] Lorenz system with negative values of parameters appears in the analysis of travelling waves in Maxwell-Bloch equations.