Mastermath	**Commutative algebra mock exam**	**3 hours**

- Write your name, university, and student number on every sheet you hand in.
- You *may* use a copy of the book and/or your notes during the exam. You may not use the internet to search for answers, or communicate with others.
- Unless stated otherwise, you need to give full proofs in all your answers. You are allowed to use results that are treated in the book and lectures.
- If you cannot do a part of a question, you may still use its conclusion later on.
- There are **5** questions in total. The exam continues on the back of this sheet.

(1) Let A be a ring. *This question is based on exercises from Chapter 1. Unless otherwise stated you should not assume the results from these exercises. You may use (without proof) that for any f in A the natural map $\mathrm{Spec}(A_f) \to \mathrm{Spec}(A)$ is an open immersion (i.e., its image $D(f) \subset \mathrm{Spec}(A)$ is open, and the map $\mathrm{Spec}(A_f) \to D(f)$ is a homeomorphism). Additionally, you may use that the sets of the form $D(f)$ form a basis for the Zariski topology on $\mathrm{Spec}(A)$.*

  (a) Let U, $V \subseteq \mathrm{Spec}\, A$ be open (in the Zariski topology). Show that $U \cap V$ is open.

  (b) Show that $\mathrm{Spec}\, A$ is compact when equipped with the Zariski topology.

  (c) If A is noetherian, and $U \subseteq \mathrm{Spec}\, A$ an open subset, show that U is compact.

  (d) Does there exist a ring A such that $\mathrm{Spec}\, A$ is homeomorphic to the closed interval $[0, 1] \subseteq \mathbb{R}$ with the euclidean topology?

(2) Let R be a ring, and I an ideal. As usual, we write (R/I)-mod for the category of (R/I)-modules. Let C denote the full subcategory of R-mod whose objects are those modules M with $\mathrm{Ann}_R(M) \supset I$.

  (a) Write down a functor from C to (R/I)-mod which is an equivalence of categories.

  (b) Prove that the functor from (a) is an equivalence.

  (c) Show that $\mathrm{Ann}_{\mathrm{R}}(I/I^2) \supset I$.

  (d) Combining parts (a) and (b), we can view I/I^2 as an (R/I)-module. Show the (R/I)-modules I/I^2 and $I \otimes_R R/I$ are isomorphic.

  (e) Now let $R = k[X, Y]/\langle XY\rangle$, and let $I = \langle X, Y\rangle$ (note that $R/I = k$). What is the dimension of $I \otimes_R R/I$ as a k-vector space?

(3) Let k be a field. At the top of the following table, two rings R, each with an R-algebra A, are listed.

	$R = k[X, Y]$, $A = R[Z]/\langle XZ - Y\rangle$	$R = k[X]$, $A = \frac{R[Y,Z]}{\langle YZ-X\rangle}$
A is a finitely generated R-algebra		
A is a finitely generated R-module		
A is a flat R-module	(b)	(c)

  (a) **Copy the above table onto your answer paper**, then fill in each box in the table with T or F, according to whether or not the given property is true for the given ring R, and R-algebra A (sometimes viewed as R-module) in that column. **You do not need to justify your answers to this part.**

  (b) Prove your answer in the box marked (b).

  (c) Prove your answer in the box marked (c).

(4) Let A be a ring, $\mathfrak{a} \subseteq A$ an ideal, and M an A-module. We equip A and M with the $\mathfrak{a}$-topology. Show that the multiplication map $A \times M \to M$ is continuous (where we

equip the first factor with the product topology). Deduce that the multiplication map $R \times R \to R$ is continuous, i.e. R is a *topological ring*.

(5) (a) Consider the $\mathbb{Z}$-module $M := \mathbb{Z}/2020\mathbb{Z}$. Compute the cardinality $\#(M \otimes \mathbb{Z}[\frac{1}{6}])$.

(b) Compute $(\mathbb{Q}/\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})$.

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