Mastermath	Commutative algebra exam	14:00-17:00 18/1/2022

- Write your name, university, and student number on every sheet you hand in.
- You may use a printout of Altman-Kleiman's book *A term of commutative algebra*.
- Motivate all your answers
- If you cannot do a part of a question, you may still use its conclusion later on.

(1) Fix a prime number p, and let $R = \mathbb{Z}_{(p)}$, the sub-ring of $\mathbb{Q}$ consisting of rational numbers which can be written as a fraction where the denominator is a power of p. We write $\mathfrak{m} = pR$, the unique maximal ideal of R.

(a) For which positive integers n is $R/\mathfrak{m}^n$ a domain (recall that positive means > 0)?

(b) For which positive integers n is $R/\mathfrak{m}^n$ an Artinian ring?

(c) Let $f\colon M \to M'$ be a map of finitely generated R-modules. Assume that the induced map $M \otimes_R R/\mathfrak{m} \to M' \otimes_R R/\mathfrak{m}$ is surjective. Show that f is surjective.

(2) Let R be a ring and A be an R-algebra. Let M be an R-module.

(a) Show that if M is a finitely generated R-module, then $M \otimes_R A$ is a finitely generated A-module.

(b) Show that if $M \otimes_R A$ is a finitely generated A-module and A is faithfully flat over R, then M is a finitely generated R-module.

(c) Show that if M is a flat R-module, then $M \otimes_R A$ is a flat A-module.

(d) Show that if $M \otimes_R A$ is a flat A-module and A is faithfully flat over R, then M is a flat R-module.

(3) Let k be a field, and $R = k[x, y]$ the polynomial ring. Let $A = R[t]/(x^2t - y^2)$.

(a) Is A finitely generated as an R-module?

(b) Is A integral as an R-algebra?

(c) Prove that A is an integral domain.

(d) Let $\mathfrak{m} = (x - 1, y) \subseteq R$ and $S = R - \mathfrak{m}$. Show that the induced map $S^{-1}R \to S^{-1}A$ is an isomorphism.

(e) What is the (Krull) dimension of A?

(4) Let $f : R \to A$ be a ring morphism and $f^* = \operatorname{Spec}(f) : \operatorname{Spec}(A) \to \operatorname{Spec}(R)$ be the induced map of prime spectra. Let M be an A-module, which we can also consider as an R-module via restriction of scalars along f.

(a) Show that $f^*(\operatorname{Ass}_A(M)) \subseteq \operatorname{Ass}_R(M)$.

(b) Let k be a field. Put $R = k$ and $A = k[x_1, x_2, \ldots]/(x_1^2, x_2^2, \ldots)$, and let $f : R \to A$ be the k-algebra map sending 1 to 1. Show that $\operatorname{Spec}(A) = \{(x_1, x_2, \ldots)\}$. Deduce that $\operatorname{Ass}_A(A) = \emptyset$ and $\operatorname{Ass}_R(A) = \{(0)\}$. Conclude that $f^*(\operatorname{Ass}_A(A)) \neq \operatorname{Ass}_R(A)$.

(c) Let $\mathfrak{p} \in \operatorname{Ass}_R(M)$ and $m \in M$ with $\operatorname{Ann}_R(m) = \mathfrak{p}$. Write $\mathfrak{a} = \operatorname{Ann}_A(m)$. Show that $\operatorname{Ass}_R(A/\mathfrak{a}) \subset \operatorname{Ass}_R(M)$.

(d) Show that f induces an injective morphism $g : R/\mathfrak{p} \to A/\mathfrak{a}$. Deduce that there exists $\bar{\mathfrak{q}} \in \operatorname{Spec}(A/\mathfrak{a})$ minimal prime of $A/\mathfrak{a}$ with $g^*(\bar{\mathfrak{q}}) = (0)$ (Hint: use localisation at $(R/\mathfrak{p}) - \{0\}$; you can use Exercise (3.16)).

(e) In this question we assume that A is Noetherian. Show that there exists $\mathfrak{q} \in \operatorname{Spec}(A)$ such that $\bar{\mathfrak{q}} = \mathfrak{q} + \mathfrak{a}$ and $\mathfrak{q} \in \operatorname{Ass}_A(A/\mathfrak{a})$ (Hint: use (17.14)). Deduce that $f^*(\operatorname{Ass}_A(M)) = \operatorname{Ass}_R(M)$.