Exam Algebraic Geometry 1

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Márton Hablicsek and Martijn Kool

- Time allowed: 3 hours. Note: the exercises continue on the next page!
- You may consult a (clean) copy of the lecture notes, but no other sources.
- Throughout, k denotes an arbitrary **algebraically closed field of characteristic 0** and all varieties are varieties over k unless stated otherwise.

Exercise 1. (1 point) Let X be an affine variety over k. Prove the following statement. There exist two non-empty disjoint closed subsets $X_1, X_2 \subset X$ such that $X = X_1 \cup X_2$ if and only if there exists an element $e \in A(X)$, not equal to the constant function 0 or 1, such that $e^2 = e$.

Exercise 2. (2 points) Let A be an Abelian group. The torsion subgroup A^t of A is defined as the subgroup of torsion elements of A, i.e., $A^t := \{a \in A : \exists n \in \mathbb{Z}_{>0}, na = 0\}$. Any homomorphism of Abelian groups $A \to B$ restricts to a homomorphism $A^t \to B^t$. Now, let X be a *finite* topological space (meaning that the topology consists of finitely many open subsets of X), and $\mathcal{F}$ a sheaf of Abelian groups on X. We define the presheaf of abelian groups $\mathcal{F}^t$ on X as

$$\mathcal{F}^t(U) = \mathcal{F}(U)^t$$

for any open subset $U \subset X$ with natural restriction morphisms (induced as above). Prove that $\mathcal{F}^t$ is a sheaf.

Exercise 3. (2 points) Let P be a point on a smooth irreducible projective curve C of genus g. Prove that any divisor D on C with $\deg(D) = 0$ is linearly equivalent to a divisor of the form $D_0 - g \cdot P$, where $D_0 \geq 0$ and $\deg(D_0) = g$. *Hint: Use Riemann-Roch.*

Exercise 4. Let $g \in \mathbb{Z}_{\geq 2}$. Consider the projective curve $C' = Z_{\mathrm{proj}}(y^2 z^{2g} - f(x,z)) \subset \mathbb{P}^2$, where

$$f(x,z) = \prod_{i=1}^{2g+2} (x - \lambda_i z),$$

and $\lambda_i \in k \setminus \{0\}$ are mutually distinct, and we use homogeneous coordinates $(x : y : z)$ on $\mathbb{P}^2$.

1. **(1 point)** Prove that $C' \subset U_1 \cup U_2$, where $U_1 = \mathbb{P}^2 \setminus Z_{\mathrm{proj}}(y)$, $U_2 = \mathbb{P}^2 \setminus Z_{\mathrm{proj}}(z)$ are standard affine charts of $\mathbb{P}^2$. Prove that $(0:1:0)$ is the only singular point of C'. *Hint: You may use without proof that $y^2 - f(x,1)$ and $z^{2g} - f(x,z)$ are irreducible polynomials. It is easiest to first consider the chart U_2.*

There exists a smooth *projective* curve C with the following properties. There are two open subsets $U, V \subset C$ covering C with the following properties. There is an isomorphism $\varphi : U \to Z(f_1) \subset \mathbb{A}^2$, where we use coordinates (x,y) on $\mathbb{A}^2$ and $f_1 = y^2 - f(x,1)$. There is an isomorphism $\psi : V \to Z(f_2) \subset \mathbb{A}^2$, where we use coordinates (v,w) on $\mathbb{A}^2$ and $f_2 = w^2 - f(1,v)$. Finally, $\varphi(U \cap V) = Z(f_1) \setminus Z(x)$, $\psi(U \cap V) = Z(f_2) \setminus Z(v)$ and on the overlap we have an isomorphism

$$\psi \circ \varphi^{-1} : Z(f_1) \setminus Z(x) \subset \mathbb{A}^2 \to Z(f_2) \setminus Z(v) \subset \mathbb{A}^2, \quad (x,y) \mapsto (1/x, y/x^{g+1}) = (v,w).$$

Informally: C is obtained by gluing the affine curves $Z(f_1)$, $Z(f_2)$ via the above isomorphism. These are all facts you do not have to prove. (Continuation on next page.)

2. **(1 point)** For any point $P = (\xi, \eta) \in Z(f_1)$ with $\eta \neq 0$, show that $\partial f_1/\partial y|_P \neq 0$. By Exercise 7.7.6, $x - \xi$ is a uniformizer at P (you do not have to reprove this). For any point $P = (\lambda_i, 0) \in Z(f_1)$, show that $\partial f_1/\partial x|_P \neq 0$. By Exercise 7.7.6, y is a uniformizer at P (you do not have to reprove this).

3. **(1 point)** Prove that the rational 1-form $(1/y)dx$ is a regular on $Z(f_1)$.

4. **(1 point)** Prove that v is a uniformizer at the points $(0, \pm 1)$ of $Z(f_2)$. Prove that $(1/y)dx$ extends to a regular 1-form on C.

5. **(1 point)** Prove that $\deg(\operatorname{div}((1/y)dx)) = 2g - 2$. This implies C has genus g.