INSTITUTE FOR THEORETICAL PHYSICS
UTRECHT UNIVERSITY

Exam "Statistical Field Theory"

Thursday, February 2nd, 2023

Duration of the exam: 3 hours

1. *Use a separate sheet for every exercise.*
2. *Write your name and initials on all sheets, and on the first sheet also your student ID number.*
3. *Write clearly, unreadable work cannot be corrected.*
4. *You are* **NOT** *allowed to use the book "Ultracold Quantum Fields" or any other book. You may use one A4 with handwritten notes.*
5. *Remember to move on to the next question if a particular question takes too much time, and if possible come back to it after you have gone through the whole exam in this manner.*

Exercise 1: Perturbation theory of the external potential.
Consider a system of non-interacting and spin-full fermions with mass m in an external potential $V_0\sigma^z$, with σ^z the Pauli matrix in the z direction. The corresponding action for the complex field $\phi_\alpha(\mathbf{x},\tau)$ is given by

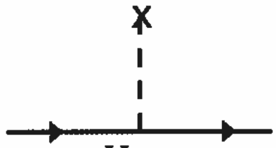
$$S[\phi^*,\phi] = \sum_{\alpha,\alpha'} \int d\tau \int d\mathbf{x}\; \phi^*_\alpha(\mathbf{x},\tau) \left\{ \left(\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2\nabla^2}{2m} - \mu \right) \delta_{\alpha\alpha'} + V_0\sigma^z_{\alpha,\alpha'} \right\} \phi_{\alpha'}(\mathbf{x},\tau)\ , \quad (1)$$

and the single-particle eigenstates $\chi^\pm_{\mathbf{n},\alpha}(\mathbf{x})$ obey the Schrödinger equation

$$\sum_{\alpha'} \left\{ -\frac{\hbar^2\nabla^2}{2m}\delta_{\alpha\alpha'} + V_0\sigma^z_{\alpha\alpha'} \right\} \chi^\pm_{\mathbf{n},\alpha'}(\mathbf{x}) = \epsilon_{\mathbf{n},\pm}\chi^\pm_{\mathbf{n},\alpha}(\mathbf{x})\ . \quad (2)$$

a) **(10 pts.)** Give the exact Green's function $G_{\alpha,\alpha'}(\mathbf{x},\tau;\mathbf{x}',\tau')$ by solving for the single-particle eigenstates and energies $\epsilon_{\mathbf{n},\pm}$.
Hint: Realize that the problem is homogeneous so you can use momentum eigenstates with momentum $\hbar\mathbf{k}$.

Now we consider $V_0\sigma^z$ as a perturbation and set up a perturbation theory in the external potential. Diagrammatically we denote the perturbation $-V_0\sigma^z/\hbar$ by the Feynman diagram



$$-\frac{V_0\sigma_z}{\hbar}$$

b) **(10 pts.)** Give also the Green's function $G_{0;\alpha,\alpha'}(\mathbf{x},\tau;\mathbf{x}',\tau')$ for the problem with $V_0 = 0$ in terms of the momentum eigenstates with momentum $\hbar\mathbf{k}$.

c) **(15 pts.)** Considering $V_0\sigma^z$ as a perturbation, give $G_{\alpha\alpha'}(\mathbf{x},\tau;\mathbf{x}',\tau')$ the Green's function of the fermions up to second order in V_0, both diagrammatically and analytically.

d) **(5 pts.)** Give the self-energy $\Sigma_{\alpha,\alpha'}(\mathbf{x},\tau;\mathbf{x}',\tau')$ and its Fourier transform $\Sigma_{\alpha,\alpha'}(\mathbf{k},i\omega_n)$.

e) **(10 pts.)** Give the exact Dyson equation for the exact Green's function, both diagrammatically and analytically, and both in coordinate space and momentum space.

f) **(10 pts.)** Show that $G_{\alpha,\alpha'}(\mathbf{x},\tau;\mathbf{x}',\tau')$ from part a) satisfied the Dyson equation from part e).

g) **(5 pts.) Bonus:** Explain how the self-energy is modified if the potential is spatially dependent such as $V_0(\mathbf{x})\sigma^z$. (No calculation is needed.)

Exercise 2: Hubbard-Stratonovich transformation for metallic ferromagnetism
The capacity of the interacting electron-system to form a ferromagnetic phase reflects the competition between the kinetic energy and the interaction potential. Being forbidden by the Pauli exclusion principle to occupy the same site, electrons of the same spin can escape the local Hubbard interaction. However, the same exclusion principle requires the system to occupy higher-lying single-particle states, raising the states' kinetic energy. When the total reduction in potential energy outweighs the increase in kinetic energy, a transition to a ferromagnetic phase is induced.

Our starting point is the usual Hubbard Hamiltonian with on-site interaction, $U \geq 0$,

$$\hat{H} = \sum_{\alpha\mathbf{k}} \xi_{\mathbf{k}} \hat{\psi}^\dagger_{\alpha\mathbf{k}} \hat{\psi}_{\alpha\mathbf{k}} + U \sum_i \hat{n}_{i\uparrow}\hat{n}_{i\downarrow},$$

where $\xi_{\mathbf{k}} = \epsilon_{\mathbf{k}} - \mu$ and

$$\hat{\psi}_{\mathbf{k}\alpha} = \frac{1}{\sqrt{N}} \sum_j e^{-i\mathbf{k}\cdot\mathbf{r}_j} \hat{\psi}_{j\alpha} \,.$$

The sum runs over N lattice sites at positions $\mathbf{r}_i$ and enumerated by i. As usual, $\hat{n}_{i\alpha} = \hat{\psi}^\dagger_{i\alpha}\hat{\psi}_{i\alpha}$ denotes the number operator for spin α on site i. (Note that we set $\hbar = 1$ throughout this exercise.)

a) i) (**5 pts.**) Show that the second term can be written as

$$H_U = \frac{U}{4}\sum_i (\hat{n}_{i\uparrow} + \hat{n}_{i\downarrow})^2 - \frac{U}{4}\sum_i (\hat{n}_{i\uparrow} - \hat{n}_{i\downarrow})^2 \,.$$

ii) (**5 pts.**) We will henceforth neglect the first term, as it does not play a role in the magnetization properties of the system. Show that the second term can be written as

$$H_U = -U \sum_i (\hat{S}^z_i)^2 \,,$$

where $\hat{S}^z_i = \frac{1}{2}\sum_{\alpha\alpha'} \hat{\psi}^\dagger_{i\alpha}\sigma^z_{\alpha\alpha'}\hat{\psi}_{i\alpha'}$.

iii) (**5 pts.**) In terms of functional integrals, show that the partition function of the system reads

$$Z = \int d[\phi^\star]d[\phi] \exp\left\{ -\int_0^\beta d\tau \left[\sum_{\mathbf{k}\alpha} \phi^\star_{\mathbf{k}\alpha}(\partial_\tau + \tilde{\xi}_{\mathbf{k}})\phi_{\mathbf{k}\alpha} - \frac{U}{4}\sum_i \left(\sum_{\alpha,\alpha'} \phi^\star_{i\alpha}\sigma^z_{\alpha\alpha'}\phi_{i\alpha'}\right)^2 \right] \right\} \,,$$

where $\phi_i(\tau)$ is a fermionic field, $\sigma^z_{\alpha\alpha'}$ is the Pauli matrix in the z direction, and $\tilde{\xi}_{\mathbf{k}} = \epsilon_{\mathbf{k}} - \tilde{\mu}$, with $\tilde{\mu}$ a shifted chemical potential. Derive this shift by considering the normal ordering of operators when setting up the functional integral.

b) (**10 pts.**) Perform the Hubbard-Stratonovich transformation to the (on average twice the local magnetization) field $m_i(\tau)$ to decouple the local quadratic interaction and show that the partition function becomes

$$Z = \int d[\phi^\star]d[\phi]d[m] \exp\left\{ -\int_0^\beta d\tau \left[\sum_{\mathbf{k}\alpha} \phi^\star_{\mathbf{k}\alpha}(\partial_\tau + \xi_{\mathbf{k}})\phi_{\mathbf{k}\alpha} - \frac{U}{2}\sum_{i\alpha\alpha'} \phi^\star_{i\alpha}\sigma^z_{\alpha\alpha'}\phi_{i\alpha'}m_i + \frac{U}{4}\sum_i m_i^2(\tau) \right] \right\} \,.$$

c) **(10 pts.)** Integrate over the fermionic degrees of freedom to obtain

$$Z = Z_0 \int d[m] \exp\left\{-\frac{U}{4}\int_0^\beta d\tau \sum_i m_i^2(\tau) + \mathrm{Tr}\left[\ln\left(1+\frac{U}{2}\sigma^z M G_0\right)\right]\right\},$$

where Z_0 and G_0 denote the quantum partition function and Green's function of the free electron gas, respectively, and the matrix $M_{i,i'}(\tau-\tau') = m_i(\tau)\delta_{i,i'}\delta(\tau-\tau')$.

d) **(10 pts.)** Obtain the effective action for this system up to second order in $m_i(\tau)$. Using $\mathrm{Tr}\ln(1-x) = -\sum_{m=1}^{\infty}\frac{1}{m}\mathrm{Tr}[x^m]$, show that its Fourier-transformed second-order term reads

$$\frac{U^2}{8}\mathrm{Tr}[\sigma^z M G_0 \sigma^z M G_0] = -\frac{U^2}{4}\sum_{\mathbf{q},n}\Pi(\mathbf{q}, i\omega_n) m_{\mathbf{q},n} m_{-\mathbf{q},-n},$$

where $\Pi(\mathbf{q}, i\omega_n)$ is introduced as the polarization tensor:

$$\Pi(\mathbf{q}, i\omega_n) \equiv -\frac{1}{N\beta}\sum_{\mathbf{k}s} G_0(\mathbf{k}+\mathbf{q}, i\omega_{s+n}) G_0(\mathbf{k}, i\omega_s)\ .$$

e) i) **(5 pts.)** Show that

$$G_0(\mathbf{k}+\mathbf{q}, i\omega_{s+n}) G_0(\mathbf{k}, i\omega_s) = \frac{1}{i\omega_n + \epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}+\mathbf{q}}}\left(G_0(\mathbf{k}, i\omega_s) - G_0(\mathbf{k}+\mathbf{q}, i\omega_{s+n})\right)\ .$$

ii) **(10 pts.)** Explicitly perform the Matsubara summation in the polarization tensor to obtain

$$\Pi(\mathbf{q}, i\omega_n) = -\frac{1}{N}\sum_{\mathbf{k}}\frac{N_{FD}(\epsilon_{\mathbf{k}}) - N_{FD}(\epsilon_{\mathbf{k}+\mathbf{q}})}{i\omega_n + \epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}+\mathbf{q}}},$$

with N_{FD} the Fermi-Dirac distribution function.

f) **(5 pts.)** Using the above results, show that the action of Z up to second order is given by

$$S^{\mathrm{eff}}[m] = \frac{1}{2}\sum_{\mathbf{q},n}\frac{U}{2}\left(1 - U\Pi(\mathbf{q}, i\omega_n)\right) m_{\mathbf{q},n} m_{-\mathbf{q},-n} + O(m^3)\ .$$