- There are 4 hours available for the problems.
- Each problem is worth 10 points.
- Be clear when using a theorem. When you are using an obscure theorem, cite a source.
- Use a different sheet for each problem.
- Clearly write DRAFT on any draft page you hand in.

MOAWOA

May 4, 2018

Problem 1. Determine all sequences $(a_1, a_2, \ldots, a_{2018})$ of positive integers such that

(i) $a_1 + a_2 + \ldots + a_{2018} = 3 \cdot 2018$;

(ii) the sum of consecutive a_i is never a power of 2. (In particular, none of the a_i is a power of 2.)

(A power of 2 is a number of the form 2^k with $k \geq 1$ an integer.)

Problem 2. Let $k > 1$ be an integer. We list all k-element subsets of $\{1, 2, \ldots, 2k-1\}$ and in each of these subsets we color one element red and one (not necessarily distinct) element blue. Our goal is to assign the colors in such a way that whenever A and B are subsets among our list with $|A \cap B| = \ell$, the red element in A differs from the blue element in B. Is this always possible

(a) if $\ell = 1$?

(b) if $\ell = 2$?

Problem 3. A real $n \times n$-matrix $A = (A_{ij})_{i,j=1}^n$ satisfies $A_{ii} = 1$ for $1 \leq i \leq n$ and $A_{ij} + A_{ji} = 1$ for $1 \leq i < j \leq n$. Show that $\det A > 0$.

Problem 4. Does there exist a smooth function $f : \mathbb{R} \to \mathbb{R}$ with the property that for all non-negative integers n the number of roots of $f^{(n+1)}$ is (strictly) greater than the number of roots of $f^{(n)}$?

Problem 5. We sample a random permutation σ of the numbers 1, 2, …, n, uniformly from the set of all $n!$ permutations. For a set $A \subset \{1, 2, \ldots, n\}$ we define the event

$$X_A = \{\, \text{all elements of } A \text{ belong to the same cycle of } \sigma \,\}.$$

Show that for any two sets S and T with at least 2 elements, the events X_S and X_T are positively correlated.

Problem 6. Determine the smallest constant $C > 0$ with the following property: if $n \geq 4$ is a positive integer, then there exist positive integers a, b, c and d such that $a + b + c + d = n$ and $\operatorname{lcm}(a, b, c, d) \leq Cn$.