MOAWOA
2018-2019

- There are 4 hours available for the problems.
- Each problem in worth 10 points. Partial solutions may be awarded partial points.
- Be clear when using a theorem. When you are using an obscure theorem, cite a source.
- Use different sheets for each problem. Write your name on every sheet you hand in.
- Clearly write DRAFT on any draft page you hand in.
- You may write answers in English or Dutch.

Problem 1 Determine all real-valued polynomials $P(x)$ such that $P(2k-1) = P(2k)$ for all k with $0 \leq k < \deg(P)$.

Problem 2 Show that for each $x, y, z \in \mathbb{N}_{>0}$ the following holds:

$$x \uparrow\uparrow y \equiv x \uparrow\uparrow z \mod 2^{\min(y,z)}$$

(The notation $a \uparrow\uparrow b$ stands for $a^{a^{\cdot^{\cdot^{\cdot^{a}}}}}$ with b times an a, for example: $2 \uparrow\uparrow 4 = 2^{2^{2^{2}}} = 65536$)

Problem 3 A triangle ABC is given. Define ℓ_1 as the median (=zwaartelijn) from B to AC and ℓ_2 as the line through A perpendicular to AC. De lines ℓ_1 and ℓ_2 intersect in D. Define E as the base of the altitude (=voetpunt van hoogtelijn) from C to ℓ_1. Suppose that $\angle BCA + \angle ABD = 90^\circ$, $|DA||AB| = |DB||BC|$ and $2|AC| = |BD|$. Show that E is the centroid (=zwaartepunt) of triangle ABC.

Problem 4 A positive integer is called a *MOAWOA*-number if every nonzero digit occurs at most twice (in decimal notation). For example, 2019, 112233 and 1000001 are *MOAWOA*-numbers but 111 and 1232123 are not. Define S as the sum of all *MOAWOA*-numbers less than 10^{10}. Show that S is divisible by $10^{10} - 1$.

Problem 5 Determine all positive integers n such that the following holds: for every $k \in \mathbb{N}$ the statement $n \mid k^n - 1$ implies that $n^2 \mid k^n - 1$.

Problem 6 *(Proposed by Julian Lyczak, IST Austria)* Let p be a prime. A subset $X \subset \mathbb{F}_p^x$ satisfies the following two properties:

- The sum $x + y$ of two distinct elements $x, y \in X$ lies in $\mathbb{F}_p^x$.
- Any element $s \in F_p^x$ can be uniquely written as the sum of two distinct elements of X.

Prove that $p = 11$ and X is either the quadratic residues modulo 11 or the quadratic non-residues.