- There are 4 hours available for the problems.
- Every problem is worth at most 10 points.
- Be clear when using a theorem. When you are using an obscure theorem, cite a source.
- Use a different sheet for each exercise.
- If you hand in a draft page clearly write DRAFT on the top of that page.

# MOAWOA
## February 10, 2012

**Problem 1.**
Does there exist a polynomial $P(z) = a_0 + a_1 z + \ldots + a_{\deg(P)} z^{\deg(P)}$ with integer coefficients such that $|a_0| + |a_1| + \ldots + |a_{\deg(P)}| > 2012$ and $|P(z)| < 2012$ for all $z \in \mathbb{C}$ with $|z| = 1$?

**Problem 2.**
(i) Let $f : (0,1) \to \mathbb{R}$ be a $\mathcal{C}^\infty$ function satisfying

$$\int_0^1 f^{(n)}(x)dx = 0 \text{ for all } n \in \mathbb{N}.$$

Does it follow that $f = 0$?

(ii) Let $f : (0,\infty) \to \mathbb{R}$ be a $\mathcal{C}^\infty$ function satisfying

$$\int_0^\infty f^{(n)}(x)dx = 0 \text{ for all } n \in \mathbb{N}.$$

Does it follow that $f = 0$?

**Problem 3.**
Prove there are infinitely many pairs $(a,b) \in \mathbb{N}^2$ with $a < b$ such that

$$1 + 2 + \ldots + a = (a+1) + (a+2) + \ldots + b.$$

*Turn page*

**Problem 4.**
Let $n \in \mathbb{N}$. We define a map $\pi$ from complex $n \times n$-matrices to real $(2n) \times (2n)$-matrices by

$$\pi(M) = \begin{pmatrix} \mathrm{Re}(M_{11}) & -\mathrm{Im}(M_{11}) & \cdots & \cdots & \mathrm{Re}(M_{1n}) & -\mathrm{Im}(M_{1n}) \\ \mathrm{Im}(M_{11}) & \mathrm{Re}(M_{11}) & \cdots & \cdots & \mathrm{Im}(M_{1n}) & \mathrm{Re}(M_{1n}) \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \mathrm{Re}(M_{n1}) & -\mathrm{Im}(M_{n1}) & \cdots & \cdots & \mathrm{Re}(M_{nn}) & -\mathrm{Im}(M_{nn}) \\ \mathrm{Im}(M_{n1}) & \mathrm{Re}(M_{n1}) & \cdots & \cdots & \mathrm{Im}(M_{nn}) & \mathrm{Re}(M_{nn}) \end{pmatrix}$$

Prove that $\det \pi(M) = |\det M|^2$.

**Exercise 5.**
Let $n \in \mathbb{N}$, $n > 3$. A knight is at the top left entry of an $n \times n$ chess board. On each entry of the board write the minimum number of moves required for the knight to reach it. Now look at the maximum of these numbers, for which $n$ is this maximum not attained in one of the corner entries of the board?

**Problem 6.**
Let $f : [0, \infty) \to \mathbb{R}$ be a twice differentiable function such that $f + f''$ is bounded. Prove there exists an $\alpha > 0$ such that $f(x) = \mathcal{O}(x^\alpha)$. Find the infimum of such $\alpha$.