- There are 4 hours available for the problems.
- Every problem is worth at most 10 points.

MOAWOA Solutions
February 10 2012

Problem 1. Does there exist a polynomial $P(z) = a_0 + a_1 z + \ldots + a_{\deg(P)} z^{\deg(P)}$ with integer coefficients such that $|a_0| + |a_1| + \ldots + |a_{\deg(P)}| > 2012$ and $|P(z)| < 2012$ for all $z \in \mathbb{C}$ with $|z| = 1$?

Solution. Yes. Take $P(z) = z^2 + 2011z - 1$. Indeed $|1| + |2011| + |-1| > 2012$ and for all $\theta \in [0, 2\pi)$

$$|P(e^{i\theta}) = |e^{i\theta}(e^{i\theta} + 2011 - e^{-i\theta})| = |2011 + 2i\sin(\theta)| = \sqrt{2012^2 + 4\sin^2(\theta) - 4023} < 2012.$$

Problem 2.
(i) Let $f : (0,1) \to \mathbb{R}$ be a $\mathcal{C}^\infty$ function satisfying

$$\int_0^1 f^{(n)}(x)dx = 0 \text{ for all } n \in \mathbb{N}.$$

Does it follow that $f = 0$?

(ii) Let $f : (0,\infty) \to \mathbb{R}$ be a $\mathcal{C}^\infty$ function satisfying

$$\int_0^\infty f^{(n)}(x)dx = 0 \text{ for all } n \in \mathbb{N}.$$

Does it follow that $f = 0$?

Solution.
(i) No. A counterexample is given by $f(x) = \cos(2\pi x)$.
(ii) No. A counterexample is given by $f(x) = e^{-1/x}$. It is clear that $f^{(n)}(x) = p_n(1/x)e^{-1/x}$ for some non-constant polynomial p_n with $p_n(0) = 0$. Thus for all $n \in \mathbb{N}$

$$\int_0^\infty f^{(n)}(x)dx = \lim_{x\to\infty} p_{n-1}(1/x)e^{-1/x} - \lim_{x\downarrow 0} p_{n-1}(1/x)e^{-1/x} = 0$$

Problem 3. (by Sjoerd Boersma)
Prove there are infinitely many pairs $(a, b) \in \mathbb{N}^2$ with $a < b$ such that

$$1 + 2 + \ldots + a = (a + 1) + (a + 2) + \ldots + b.$$

Solution 1. (by Sjoerd Boersma)
We find the solution $(2, 3)$. Now (x, y) is a solution if and only if $\frac{x(x+1)}{2} = \frac{y(y+1)}{2} - \frac{x(x+1)}{2}$, or equivalently $2x(x + 1) = y(y + 1)$. If (x, y) is a solution, $(a, b) = (3x + 2y + 2, 4x + 3y + 3)$ is also a solution, since:

$$2a(a + 1) = 2(3x + 2y + 2)(3x + 2y + 3) = 18x^2 + 24xy + 8y^2 + 30x + 20y + 12 =$$

$$16x^2 + 24xy + 9y^2 + 28x + 21y + 12 = (4x + 3y + 3)(4x + 3y + 4) = b(b + 1).$$

This way we can create a new solution from any other, and since $3x + 2y + 2 > x$ for natural x and y, there are solutions with arbitrarily large x, and their number is infinite.

Solution 2. Let $\alpha = 2a + 1$ and $\beta = 2b + 1$. Then our equation can be rewritten as $\beta^2 - 2\alpha^2 = -1$. We notice that $\beta + \sqrt{2}\alpha = 1 + \sqrt{2}$ is a solution (though not of our original equation). However the ($\mathbb{Z}[\sqrt{2}]$-)norm of $1 + \sqrt{2}$ is -1 and by the multiplicative property of the norm so is the norm of every odd power of $1 + \sqrt{2}$. We conclude that other solutions are given by

$$\beta + \sqrt{2}\alpha = (1 + \sqrt{2})^{2n+1} = \sum_{k=0}^{n} \binom{2n+1}{2k} 2^k + \sqrt{2} \sum_{k=0}^{n} \binom{2n+1}{2k+1} 2^k.$$

Thus we find a set of solutions

$$(a, b) = (n + \sum_{k=0}^{n-1} \binom{2n+1}{2k+3} 2^k, \sum_{k=0}^{n-1} \binom{2n+1}{2k+2} 2^k) \text{ with } n \in \mathbb{N}.$$

This set is obviously unbounded and therefore yields infinitely many solutions.

Problem 4.
Let $n \in \mathbb{N}$. We define a map π from complex $n \times n$-matrices to real $(2n) \times (2n)$-matrices by

$$\pi(M) = \begin{pmatrix} \mathrm{Re}(M_{11}) & -\mathrm{Im}(M_{11}) & \cdots & \cdots & \mathrm{Re}(M_{1n}) & -\mathrm{Im}(M_{1n}) \\ \mathrm{Im}(M_{11}) & \mathrm{Re}(M_{11}) & \cdots & \cdots & \mathrm{Im}(M_{1n}) & \mathrm{Re}(M_{1n}) \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \mathrm{Re}(M_{n1}) & -\mathrm{Im}(M_{n1}) & \cdots & \cdots & \mathrm{Re}(M_{nn}) & -\mathrm{Im}(M_{nn}) \\ \mathrm{Im}(M_{n1}) & \mathrm{Re}(M_{n1}) & \cdots & \cdots & \mathrm{Im}(M_{nn}) & \mathrm{Re}(M_{nn}) \end{pmatrix}$$

Prove that $\det \pi(M) = |\det M|^2$.

Solution 1. First we show that π is actually a homomorphism. For example

$$\begin{aligned}(\pi(M)\pi(N))_{2i,2j} &= \sum_{k=1}^{n} (\pi(M)_{2i,2k}\pi(N)_{2k,2j} + \pi(M)_{2i,2k-1}\pi(N)_{2k-1,2j}) \\ &= \sum_{k=1}^{n} (\mathrm{Re}(M_{ik})\mathrm{Re}(N_{kj}) - \mathrm{Im}(M_{ik})\mathrm{Im}(N_{kj})) \\ &= \mathrm{Re}\left(\sum_{k=1}^{n} M_{ik}N_{kj}\right) = (\pi(MN))_{2i,2j}.\end{aligned}$$

A similar argument can be used for the other three cases.
Write M in Jordan normal form, i.e. $M = U^{-1}JU$ for some invertible matrix U and a Jordan block matrix J. By the homomorphism property it then follows that

$$\det \pi(M) = \det(\pi(U)^{-1}\pi(J)\pi(U)) = \det \pi(J) = \prod_{i=1}^{n} \det \begin{pmatrix} \mathrm{Re}(J_{ii}) & -\mathrm{Im}(J_{ii}) \\ \mathrm{Im}(J_{ii}) & \mathrm{Re}(J_{ii}) \end{pmatrix} = \prod_{i=1}^{n} |J_{ii}|^2 = |\det M|^2.$$

Solution 2. First suppose M is diagonalizable. Then we obtain a set of linearly independent eigenvectors $v_1, v_2, \ldots, v_n$ with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ respectively. Define the vectors $V_1, V_2, \ldots, V_n \in \mathbb{C}^{2n}$ by $(V_j)_{2k-1} = (v_j)_k$ and $(V_j)_{2k} = -i(v_j)_k$. Then $\pi(M)V_j = \lambda_j V_j$. Also, since $\pi(M)$ is a real matrix, we get $\pi(M)\overline{V_j} = \overline{\lambda_j}\,\overline{V_j}$. Let us prove that these vectors $V_1, \ldots, V_n, \overline{V_1}, \ldots, \overline{V_n}$ are linearly independent. Suppose there exist $a_1, a_2, \ldots, a_{2n} \in \mathbb{C}$ such that

$$\sum_{j=1}^{n}(a_j V_j + a_{n+j}\overline{V_j}) = 0.$$

This corresponds to

$$\sum_{j=1}^{n}(a_j v_j + a_{n+j}\overline{v_j}) = \sum_{j=1}^{n}(-ia_j v_j + ia_{n+j}\overline{v_j}) = 0$$

which in turn yields

$$\sum_{j=1}^{n} a_j v_j = \sum_{j=1}^{n} a_{n+j} \overline{v_j} = 0.$$

Since the original eigenvectors $v_1, v_2, \dots, v_n$ are independent this implies $a_1 = a_2 = \dots = a_n = 0$ and $\overline{a_{n+1}} = \overline{a_{n+2}} = \dots = \overline{a_{2n}} = 0$. We conclude that $\pi(M)$ can be diagonalized with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n, \overline{\lambda_1}, \overline{\lambda_2}, \dots, \overline{\lambda_n}$ on its diagonal. Hence $\det \pi(M) = \lambda_1 \lambda_2 \cdots \lambda_n \overline{\lambda_1}\ \overline{\lambda_2} \cdots \overline{\lambda_n} = |\det M|^2$.
Now let M be a general complex $n \times n$-matrix. It is wellknown that the diagonalizable complex $n \times n$-matrices are dense in the complex $n \times n$-matrices. Hence we find a sequence $(M_k)_{k \in \mathbb{N}}$ of diagonalizable complex $n \times n$-matrices converging to M. Since the determinant and π are continuous functions:

$$\det \pi(M) = \lim_{k \to \infty} \det \pi(M_k) = \lim_{k \to \infty} |\det M_k|^2 = |\det M|^2.$$

Exercise 5. (by Sjoerd Boersma)
Let $n \in \mathbb{N}$, $n > 3$. A knight is at the top left entry of an $n \times n$ chess board. On each entry of the board write the minimum number of moves required for the knight to reach it. Now look at the maximum of these numbers, for which n is this maximum not attained in one of the corner entries of the board?

Solution. (by Sjoerd Boersma)
Only when $n \equiv 1 \mod 3$ the maximum is not attained in one of the corner entries. We denote the entry in row a from above and column b from the left by $[a, b]$. The knight thus starts at $[1, 1]$. We denote a move from $[a, b]$ to $[c, d]$ by $(c-a, d-a)$. Thus the possible moves for the knight are: $(-2, -1)$, $(-2, 1)$, $(-1, -2)$, $(-1, 2)$, $(1, -2)$, $(1, 2)$, $(2, -1)$, $(2, 1)$.
Now let $f_n([a, b])$ be the minimal number of moves the knight needs to reach $[a, b]$ from $[1, 1]$ on the $n \times n$-chessboard. If we make a table of the values of f_n on the chessboard for small values of n, we get:

$n = 4$

0	3	2	$\underline{5}$
3	4	1	2
2	1	4	3
$\underline{5}$	2	3	2

0	3	2	3	2
3	4	1	2	3
2	1	4	3	2
3	2	3	2	3
2	3	2	3	$\underline{4}$

$n = 5$

$n = 6$:

0	3	2	3	2	3
3	4	1	2	3	4
2	1	4	3	2	3
3	2	3	2	3	4
2	3	2	3	4	3
3	4	3	4	3	$\underline{4}$

0	3	2	3	2	3	4
3	4	1	2	3	4	3
2	1	4	3	2	3	4
3	2	3	2	3	4	3
2	3	2	3	4	3	4
3	4	3	4	3	4	$\underline{5}$
4	3	4	3	4	$\underline{5}$	4

$n = 7$

We see that the maximum is attained in a corner for $n = 4, 5, 6$ and not for $n = 7$. In particular, for $n = 5, 6$ the maximum is attained at $[n, n]$.

If $n \equiv 1 \mod 3$, let $n = 3m + 1$. Then it is possible to get from $[1, 1]$ to $[n, n]$ in $2m$ moves, namely $m \times (1, 2)$ and $m \times (2, 1)$. If m is even, $[1, n]$ can be reached in $2m$ moves, namely $\frac{m}{2} \times (1, 2)$, then $\frac{m}{2} \times (-1, 2)$, then $\frac{m}{2} \times (2, 1)$ and finally $\frac{m}{2} \times (-2, 1)$. Similarly $[n, 1]$ can be reached in $2m$ moves. If m is odd and not equal to 1, let $m = 2k + 1$. Then $[1, n]$ can be reached in $2m - 1 = 4k + 1$ moves, namely $(k + 2) \times (1, 2)$, then $k \times (-1, 2)$, then $(k - 1) \times (2, 1)$ and finally $k \times (-2, 1)$. It follows that for all corners c of the board $f_n(c) \leq 2m$ if $n \equiv 1 \mod 3$ and $n > 4$. However $[n, n - 1]$ cannot be reached within $2m$ moves: let for $[a, b]$ on the chessboard $g([a, b]) = a + b$. If you move from one square to another in only one move, the value of g will decrease 3, decrease 1, increase 1 or increase 3. If we go from $[1, 1]$ tot $[n, n - 1]$, g has to increase $2n - 3 = 6m - 1$. Thus we need at least ceiling of $\frac{6m-1}{3}$ moves, so at least $2m$. However in $2m - 1$ moves the value of g can change with either of $-6m + 3$, $-6m + 5$, $-6m + 7$, $\ldots$, $6m - 3$. Thus a change of $6m - 1$ is not possible in $2m$ moves, and $f_n([n, n - 1]) > 2m$. Hence the maximum is not attained in one of the corners.

For $n = 5$, the maximum is attained at $[n, n]$ and has value 4. Given a value $n \equiv 2 \mod 3$, let $n = 3k + 2$ and suppose the maximum is attained at $[n, n]$ where $f_n([n, n]) = 2k + 2$ (as for $n = 5$). On the left upper $n \times n$-area on the $(n + 3) \times (n + 3)$-chessboard the squares can be reached in the same way as on the $n \times n$-chessboard, and thus $f_{n+3}([a, b]) \leq f_n([a, b]) \forall a, b \leq n$. All new squares can be reached in at most two moves from the left upper $n \times n$-area, and thus $f_{n+3}([a, b]) \leq 2k + 4 \ \forall a, b$. $[n+3, n+3]$ can now be reached in exactly $2k+4$ moves by doing moves $(1, 2)$ and $(2, 1)$ after going to $[n, n]$. $g([n+3, n+3]) - g([1, 1]) = 2n+4 = 6k+8$. thus it is not possible to reach $[n+3, n+3]$ in less than $\frac{6k+8}{3}$ moves and thus at least $2k + 3$ moves are necessary. However, since the knight alternates black and white squares actually at least $2k + 4$ moves are required. Thus the maximum is attained in the lower right corner and $f_{3(k+1)+2}([3(k + 1) + 2, 3(k + 1) + 2]) = 2(k + 1) + 2$. By induction this is now true for all $n \equiv 2 \mod 3$, $n \geq 5$.

The proof for $n \equiv 0 \mod 3$ works almost equivalent. Let here $n = 3k$ and the maximum is attained at $[n, n]$ with value $2k$. This is true for $n = 6$. This time on the $(n + 3) \times (n + 3)$-chessboard no value of f_{n+3} is larger than $2k + 2$. To reach $[n + 3, n + 3]$ takes at least $\frac{2n+4}{3}$ and thus $2k + 2$ moves. This is possible if we again move $(1, 2)$ and $(2, 1)$ from $[n, n]$. Thus the maximum is attained at the lower left corner and equals $2(k+1)$. By induction it is now true for every threefold larger than 3.

Exercise 6. Let $f : [0, \infty) \to \mathbb{R}$ be a twice differentiable function such that $f + f''$ is bounded. Prove there exists an $\alpha > 0$ such that $f(x) = \mathcal{O}(x^\alpha)$. Find the infimum of such α.

Solution. Without loss of generality we may assume that $f(0) = f'(0) = 0$, because we may substract terms like $f(0)\cos(x)$ and $f'(0)\sin(x)$ from f without changing the problem. Now let $x > 0$. With the mean value theorem we find a $\xi \in (0, x)$ such that

$$f(x)^2 + f'(x)^2 = x|2f'(\xi)f(\xi) + 2f''(\xi)f'(\xi)| \leq 2Cx|f'(\xi)| \leq 2Cx \max_{y\in[0,x]} |f'(y)|$$

for some constant $C > 0$ that bounds $f + f''$. We see that

$$\left(\max_{y\in[0,x]} |f'(y)|\right)^2 \leq \max_{y\in[0,x]} (f(y)^2 + f'(y)^2) \leq 2Cx \max_{y\in[0,x]} |f'(y)|$$

from which it follows that $f'(x) = \mathcal{O}(x)$ and thus

$$f(x)^2 \leq f(x)^2 + f'(x)^2 \leq 2Cx \max_{y\in[0,x]} |f'(y)| = \mathcal{O}(x^2).$$

We conclude that $\alpha = 1$ works. This is actually the infimum which is seen by taking the function $f(x) = x\sin(x)$. Then $f(x) = \mathcal{O}(x^\alpha)$ only when $\alpha \geq 1$ and $|f(x) + f''(x)| = |2\cos(x)| \leq 2$.

Remark. The fist part of the problem can be solved differently.
Let $x > 0$. One easily proves using the mean value theorem that there exist $0 < \mu, \nu < x$ such that for any differentiable function $g : [0, \infty) \to \mathbb{C}$, satisfying $g(0) = 0$, we have

$$|g(x)| \leq |\mathrm{Re}(g(x))| + |\mathrm{Im}(g(x))| = x(|\mathrm{Re}(g'(\mu))| + |\mathrm{Im}(g'(\nu))|) \leq 2x \max_{y\in[0,x]} |g'(y)|.$$

Henceforth (assuming $f(0) = f'(0) = 0$)

$$\begin{aligned} |f(x)| = |e^{-ix}f(x)| &\leq 2x \max_{y\in[0,x]} |e^{-iy}(f'(y) - if(y))| = 2x \max_{y\in[0,x]} |e^{iy}(f'(y) - if(y))| \\ &\leq 2x \max_{y\in[0,x]} 2y \max_{z\in[0,y]} |e^{iz}(f''(z) + f(z))| = \mathcal{O}(x^2) \end{aligned}$$

and we conclude that $\alpha = 2$ works.